# GEOMETRICAL OPTICS

**Geometric Optics** is the study of light energy and its properties. Light is a form of energy which stimulates the sense of sight. This is evident in the following aspects.

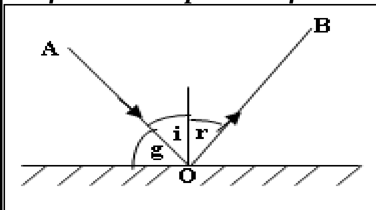
- When light enters our eyes, it enables us to see.
- When light is absorbed by green plants, it enables them make their own food for their growth.
- When light is incident on certain metals, the surface electrons acquire energy and escape. This process is called Photo electric emission.

**A ray of light;** Is the path taken by light energy. A ray is represented by a straight light line with an arrow to show the direction taken by light energy.

## (a) REFLECTION AT PLANE SURFACES

Reflection is the changing of direction of light after striking a shiny surface.

***Reflection at a plane surface***




Ray AO is the incident ray, OB is the reflected ray, ON is the normal line.
g is the glancing angle.
**i** is angle of incidence
**r** is angle of reflection.

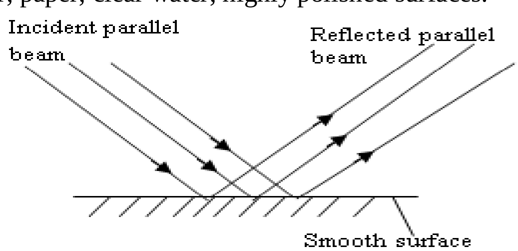
**Laws of reflection**

1. *The incident ray, the reflected ray and the normal at the point of incidence all lie on the same plane.*
2. *the angle of incidence is equal to the angle of reflection ie angle **i**= angle **r***

**Types of reflection**

***(i) Regular reflection***

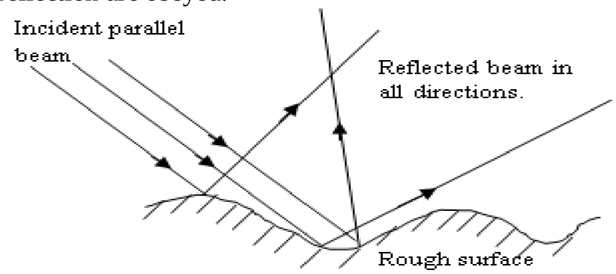
Is reflection where an incident parallel beam is reflected as a parallel beam when light falls on a smooth surface e.g. plane mirror, paper, clear water, highly polished surfaces.



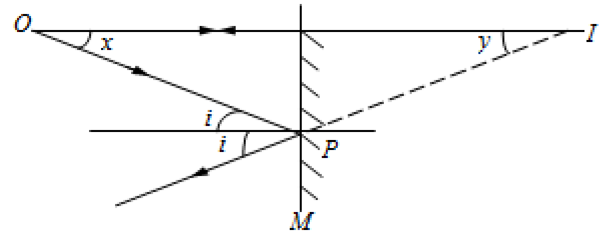

Light is said to have undergone a regular reflection

***(ii) Irregular reflection or diffuse reflection.***

Is reflection where an incident parallel beam is reflected in different directions when light falls on a rough surface e.g. iron sheets, unclear water, etc. At each point the laws of reflection are obeyed.




**Formation of images by a plain mirror**




Consider an object O placed in front of a plane mirror. Rays of light form O are reflected form the mirror and appear to come form I. I is the virtual image of O.

$< i =< r$ ($laws\ of\ reflection\ of\ light$)
$< i =< x$ ($alternate\ angles$)
$< r =< y$ ($corresponding\ angles$)
Thus, $<\boldsymbol{x} = <\boldsymbol{y}$

Since side MP is common to both triangles $\Delta OMP$ and $\Delta IMP$, the triangles are congruent. Hence $OM = MI$.

**<u>Characteristics of images formed in a plane mirror.</u>**

- Virtual
- Erect(upright)
- Laterally inverted
- Same size as object
- Same distance behind the mirror as the object is in front. The line joining the object and the image meet the mirror at $90^0$.

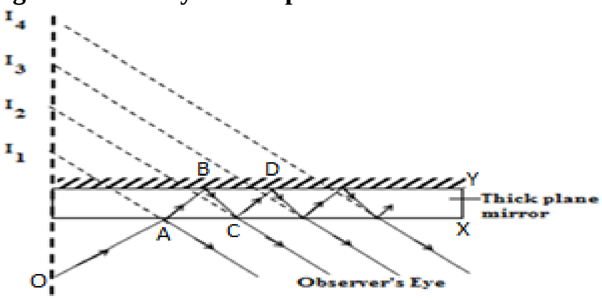
**Question:**

Explain why when you look in a mirror, you are able to see the image of your face but when you look onto the wall, you cannot see your image.

**Answer:**

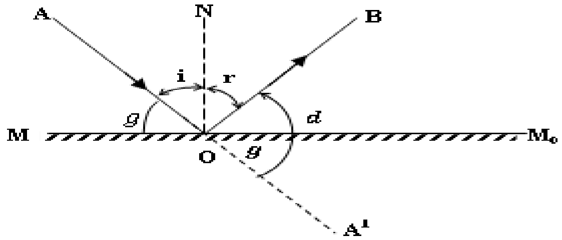
- *The plane mirror is highly polished, thus parallel light rays from the face strike the mirror surface and are reflected as a parallel beam hence forming the image.*
- *However, has some irregularities and so a parallel incident beams from the face is not reflected as a parallel beam hence the image cannot be seen.*

**Image formation by a thick plane mirror**




- When light falls on a thick plane mirror, part of it will be reflected at the upper surface, (X) and part will be transmitted.
- The transmitted light then under goes reflection at the lower surface, (Y) of the thick mirror.
- The light ray reflected at the upper surface forms its image, ($I_1$) and that which is transmitted also gives its image.
- At C both reflection and refraction occur. The refracted light leads to the formation of image ($I_2$).
- These successive total internal reflections bring about the formation of multiple images in thick plane mirrors. Thus, several images (**ghost images**) are formed by reflection in a thick plane mirror.

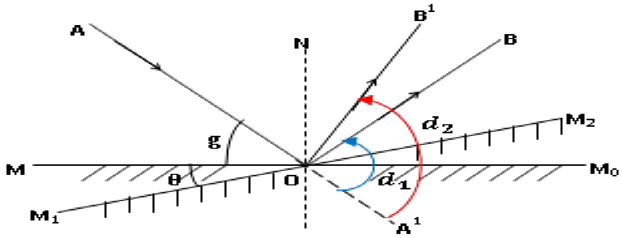
***Deviation of light at a plane surface***




| | |
|---|---|
| Angle of Deviation, d;<br>d = Angle $A^1OB$<br>d = $g$ + Angle $M_0OB$<br>d = $g + (90 - r)$<br><br>But $i$ = r<br>(From the law of reflection). | d = $g + (90 - i)$, But;<br>$(90 - i) = g$<br>(*Vertically opposite angles*)<br>d = $g + g$<br>$\boldsymbol{d} = 2\boldsymbol{g}$<br>*Angle of deviation produced at plane surface is equal to twice the glancing angle.* |

**Deviation of reflected ray by rotating the mirror keeping the incident ray fixed.**

(Relationship between the angle of rotation of a mirror and angle of rotation of the reflection ray).




MMo is the initial position of the mirror. Keeping the direction of the incident ray constant, the mirror is rotated about O through an angle ϴ to a position $M_1M_2$

OB is the reflected ray in position MMo of the mirror and $OB^1$ is the reflected ray in position $M_1M_2$ of the mirror.

When mirror is in position MMo; the glancing angle is g.

Deviation produced by the mirror in position, MMo is twice the glancing angle;

$$d_1 = \text{Angle BO } A^1 = 2g \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (i)$$

When mirror is rotated to position $M_1M_2$, and the glancing angle Glancinng ngle , $g^1 = g + \theta$

Deviation produced by mirror in position $M_1M_2$, is twice the glancing angle.

$$d_2 = \text{Angle } B^1\text{O } A^1 = 2(g + \theta) \dots\dots\dots\dots\dots (ii)$$

Angle of rotation of reflected ray = Angle $B^1$O B

But;

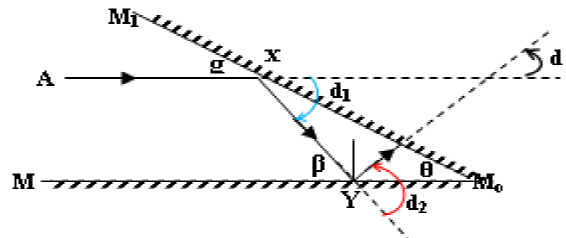
$$\text{Angle } B^1\text{O B} = \text{Angle } B^1\text{O } A^1 - \text{AngleBO } A^1$$

$$\text{Angle } B^1\text{O B} = 2(g + \theta) - 2g$$

$$\textbf{Angle } \boldsymbol{B^1}\textbf{O B} = 2\boldsymbol{\theta}$$

*Therefore, the reflected ray is rotated through an angle which is twice the angle of rotation of the mirror. This is applied in optical mirror galvanometers to measure very small currents and voltages.*

***Deviation produced by successive reflection of two inclined mirror***




Consider two mirrors $M_1M_o$ and $MM_o$ inclined at angle ϴ. Consider a ray AX incident to the mirror $M_1M_o$ at a glancing angle of g.

| | |
|---|---|
| Deviation produced at X:<br>$d_1 = 2g$ (*Clockwise*) … ….($i$)<br><br>Deviation produced at Y;<br>$d_2 = 2\beta$ (*Anticlocwise*) … ($ii$) | Net deviation produced by the two mirrors;<br>$d = d_2 - d_1$<br>$d = 2\beta - 2g$<br>$d = 2(\beta - g)$ |

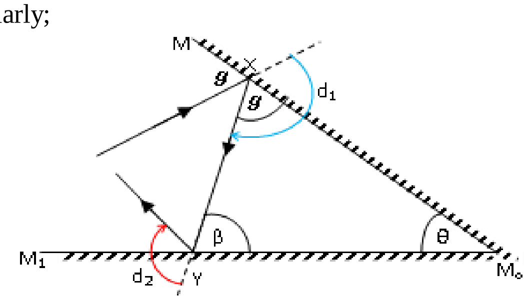
*But; $\beta = g + \theta$ ;*

*(Sum of two opposite interior angles is equal to one exterior angle)*

$$d = 2[(g + \theta) - g]$$

$$\boldsymbol{d} = 2\boldsymbol{\theta} \quad (Anticlockwise)$$

Similarly;



| Deviation produced at X: $d_1 = 2g\ (Clockwise)\ldots(i)$ <br><br> Deviation produced at Y; $d_2 = 2\beta\ (Clocwise)\ldots(ii)$ | Net deviation produced by the two mirrors; <br> $d = d_2 + d_1$ <br> $d = 2\beta + 2g$ <br> $d = 2(\beta + g)\ \ldots\ldots\ldots.(iii)$ |
|---|---|

But; From triangle $XYM_0$; $\beta + g + \theta = 180^0$;
(Interior Angle sum of a triangle)
$(\beta + g) = 180^0 - \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..\ldots(iv)$

Put Equation (iv) into Equation (iii)

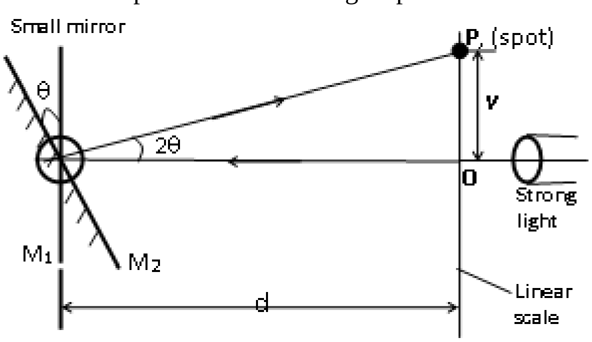
$$d = 2(180^0 - \theta)$$
$$d = [360^0 - 2\theta]\quad (Clockwise).$$
$$\boldsymbol{d} = 2\boldsymbol{\theta}\quad (Anticlockwise)$$

## Applications of Rotation of mirrors

**(i) Principal of an optical lever in mirror galvanometers.**

It uses a beam of light known as the optical lever to create a bright spot which acts as a pointer that moves on a linear scale. It is used to measure or detect small currents.

- Before the currents are passed through the cell, the small mirror attached to the coil axis is in position **$M_1$**. The beam of light from the source is normal to the mirror. The light which is reflected makes a bright spot at the initial position, **O** of the linear scale.
- When the current to be measured is passed through the coil, the coil turns and rotates the mirror through the coil to another position **$M_2$**. The light spot is then deflected to another position **P**. The value of the current passed can be read at the position P of the bright spot.




$$\tan 2\boldsymbol{\theta} = \frac{\boldsymbol{y}}{\boldsymbol{d}}$$
$$\boldsymbol{y} = \boldsymbol{d}\tan 2\boldsymbol{\theta}$$

But for small angles in radians, $\tan 2\theta \approx 2\theta$

$$\boldsymbol{y} = 2\boldsymbol{\theta}\boldsymbol{d}$$

The sensitivity of the galvanometer is the deflection $\theta$ per unit current passed through the galvanometer.

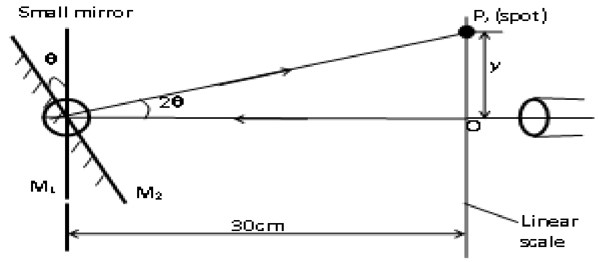
$$\text{Sensitivity, } S = \frac{\theta}{I};$$

Normally given as radians per milliampere.

$$\boldsymbol{\theta} = \boldsymbol{SI}$$

**Example**

An optical galvanometer of sensitivity 0.05 radians per mA is used to measure current of 0.2 mA. The distance of the cell from the linear scale 30cm. Find the displacement of the light spot from the initial position.

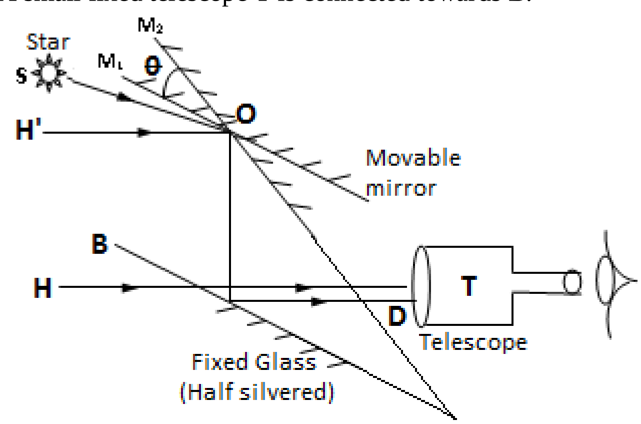



**Solution**

| Sensitivity, <br> $S = \frac{\theta}{I} = 0.05 rads\ per\ mA$ <br> 1mA turns the coil through 0.05radians. <br> $1mA \rightarrow 0.05rads$ <br> $0.2mA \rightarrow \theta$ <br> $\theta = 0.2 \times 0.05\ rads$ <br> $\theta = 0.1\ rads$ <br> The reflected ray is turned through an angle of $2\theta$. | Thus; <br> $2\theta = 2 \times 0.1 = 0.2rads.$ <br> $\tan 2\theta = \frac{y}{d}$ <br> $\tan 0.2 = \frac{y}{30}$ <br> $y = 30\tan 0.2$ <br> But for small angles in radians, $\tan 2\theta \approx 2\theta$ <br> $y = 30(0.2)$ <br> $y = 6cm$ |
|---|---|

**(ii) Principal of the sextant**

The sextant is an instrument used in navigation for measuring the angle of elevation of the sun or stars. It consists essentially of a fixed glass B, silvered on a vertical half, and a silvered mirror O which can be rotated about a horizontal axis.
A small fixed telescope T is connected towards B.




Suppose that the angle elevation of the sun, S is required.

- The setup is as above. B is half silvered fixed mirror while m can rotate.
- First rotate m until the image of horizon H[1] is seen to coincide with horizon H.
- At this point m and B are parallel, note position of mirror.
- M is now rotated until the image of the star is seen to coincide with the horizon H.
- The angle of rotation $\theta$ is measured. The angle of deviation is **$2\boldsymbol{\theta}$.**

**Note:** *Since the angle of deviation after two successive reflections is independent of the angle of incidence on the first mirror, the image of the sun S through T will continue to be seen on the horizon once O is adjusted, no matter how the ship pitches or rolls. This is an advantage of the sextant.*

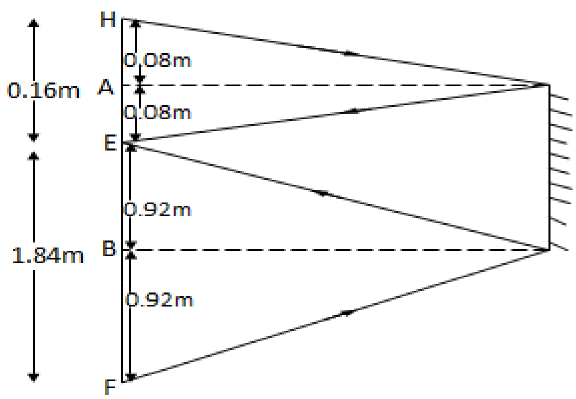
**A real Image**: Is the image formed by actual intersection of (reflected or refracted) light rays.
It can be formed on a screen.

**A virtual Image**: Is the image formed by apparent intersection of (reflected or refracted) light rays.
It cannot be formed on a screen.

**Minimum vertical length of a plane mirror**

A man 2m tall whose eye level is 1.84m above the ground looks at his image in a Vertical mirror. What must be the minimum vertical length of the mirror so that the man can see the whole of himself **completely** in the mirror.

**Solution:**




Rays from the top of the man are reflected from the top of the mirror and are incident in the man's eyes (point E is the man's eye level). Since **HA** = **AE** then,

$$AE = \frac{1}{2} \times 0.16 = 0.08\ m$$

**Similarly, $EB = BF$. Thus:**

$$EB = \frac{1}{2} \times 1.84 = 0.92\ m$$

**The minimum length of the mirror = $AE + EB$**

$$0.08 + 0.92 = 1m$$

**Hence the minimum length of the mirror is half the height of the object**

**Comparisons of Plane Mirrors and Reflecting Prisms.**

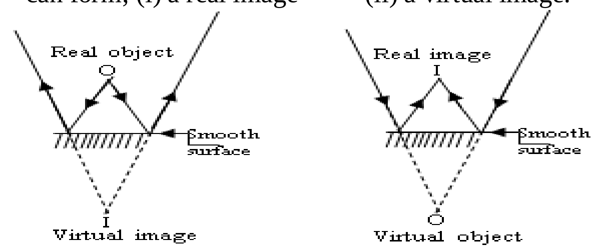
- Un like in prisms, plane mirrors produce multiple images
- The silvering in plane mirrors wears out with time while no silvering is required in prisms.
- Unlike in prisms, plane mirrors exercise loss of brightness when reflection occurs at its surface.

**EXERCISE: 1**

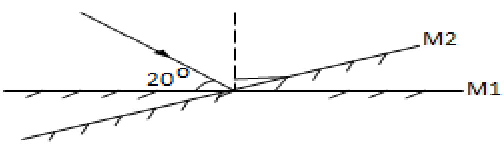
**1.** State the properties of images produced by the plane mirrors.

**2.** Two plane mirrors are inclined at an angle of $(90+\beta)^0$ to each other. Where β is a small positive angle?
 (i) Using the laws of reflection, show that a ray of light incident on any of these plane mirrors, experiences a deviation of $(180+2\beta)^0$.
 (ii) With the aid of a ray diagram, describe one application of this system when β vanishes.

**3.** With the aid of a ray diagram, show how a plane mirror can form; (i) a real image (ii) a virtual image.




**4.** What is meant by reflection of light?
**5.** State the laws of reflection of light
**6.** Distinguish between regular and diffuse reflection of light
**7.** Show with the aid of a ray diagram that when a ray of light is incident on a plane.
**8.** mirror, the angle of deviation of a ray by the plane surface is twice the glancing angle.
**9.** Derive the relation between the angle of rotation of a plane mirror and the angle of deflection of a reflected ray, when the direction of the incident ray is constant.
**10.** An incident ray of light makes an angle of 20° with the plane mirror in position m1,as shown below.




Calculate the angle of reflection, if the mirror is rotated through 6° to position M2 while the direction of the incident ray remains the same.

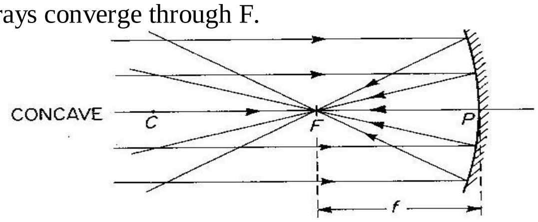
**11.** (i) Show that an incident ray of light reflected successively from two mirrors inclined at an angle $\theta$ to each other is deviated through an angle $2\theta$.
(ii) Name one application of the result in 10 (i) above.
**12.** Describe how a sextant is used to determine the angle of elevation of a star.
**13.** Show that the image formed in a plane mirror is as far behind the mirror as the object is in front.
**14.** (i) What is meant by No parallax method as applied to location of an image? (ii) Describe how the position of an image in a plane mirror can be located.
**15.** Show that for a man of height, H, standing upright the minimum length of a vertical plane mirror in which he can see the whole of him self completely is $\frac{1}{2}H$ .
**16.** With the aid of a ray diagram, explain how a thick plane mirror forms multiple images of an object. Give three reasons for using prisms rather than plane mirrors in reflecting optical instruments.

## REFLECTION AT CURVED MIRRORS

There are two types of curved mirrors namely:
1. Concave mirror / converging mirror.
2. Convex / diverging mirror.

### Action of a concave mirror

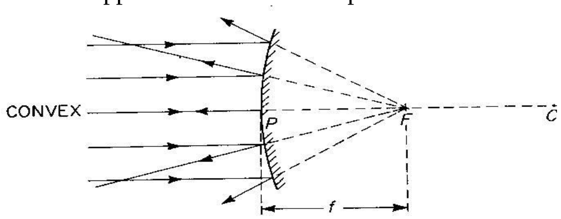
It is formed by silvering the outside part of a spherical surface.
Consider a parallel beam of light incident on a concave mirror and close to the principal axis of the mirror, after reflection the rays converge through F.




Point F is called the principal focus of the mirror since light actually passes through it.

### Action of a convex mirror

It is formed by silvering the inside part of a spherical surface.
A narrow beam of rays parallel and near to the principal axis falling on the convex mirror is reflected to form a divergent beam and appears to come from a point F behind the mirror.




The point F is the principal focus of the convex mirror and it is a ***virtual focus*** hence for a convex mirror, $f = -\mathbf{f}$ and $r = -\mathbf{r}$.
C is the centre of curvature and it is the centre of the sphere of the mirror is part. P is the pole of the mirror and it is the centre of the mirror surface.
AB is the aperture of the mirror.
The line joining C to P is called the principal axis.
The distance CP is called the radius of curvature.

### Definition of terms used in curved mirrors.

**Pole, P:** It is the mid-point or centre of the mirror. It is where the principle axis meets the mirror.

**Principal axis:** Is the imaginary line joining the pole of the mirror to the centre of curvature.

**Principal focus, F:** The principal focus of *a concave mirror* is the point through which all rays parallel and close to the principal axis converge after reflection.
The principal focus of *a convex mirror* is the point through which all rays parallel and close to the principal axis appear to diverge from after reflection.

**Focal length; f:** Focal length is the distance between the pole and principal focus. It is positive for a concave mirror and negative for a convex mirror.

**Centre of curvature:** Is the centre of the sphere of which the mirror forms part. OR It is a point on the principle axis where any ray through it hits the mirror at right angles.

**Radius of curvature:** Is the radius of the sphere of which the mirror forms part. OR It is the distance between the centre of curvature and the pole of the mirror.

**Secondary axis:** line through the centre of a thin lens or through the centre of curvature of a concave or convex mirror other than the principal axis of the lens or mirror

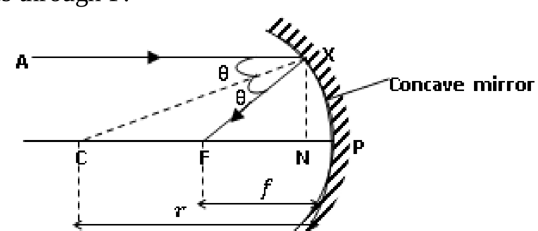
**Secondary axes:** These are rays of light that are parallel to the principle axis. They are grouped into two:
(i). Marginal rays: These are rays which are parallel and far away from the principle axis.
(ii). Central rays: These are rays which are parallel and close to the principle axis.

**Paraxial rays:** Rays which are close to principal axis and make small angles with it are called paraxial rays.

### Relationship between the radius of curvature and the focal length.

**(i) For a concave mirror**

Consider a ray parallel and near the principal axis of a converging mirror, concave mirror after reflection the ray passes through F.




If a normal line is drawn from X, it will pass through point C.

$$< AXC =< CXF = \theta \text{ (Laws of reflection)}$$
$$< AXC =< XCP = \theta \text{ (Alternating angles)}$$
$$< XFP =< AXF = 2\theta \text{ (Alternating angles)}$$

For paraxial rays such as AX and small angles in radians,
$\tan(CXN) =< CXN$, and $\tan(CXN) =< CXN$

From triangle XCN
$$\tan CXN = \tan\theta = \frac{XN}{CN} \approx \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$
From triangle XFN
$$\tan XFN = \tan 2\theta = \frac{XN}{FN} \approx 2\theta \ldots\ldots\ldots\ldots\ldots.. (ii)$$

From equations (i) and (ii);
$$\frac{XN}{FN} = 2\left(\frac{XN}{CN}\right) \Leftrightarrow CN = 2FN$$

Since$<$ **XCN, and $<$ XFN are small angles in radians**,
X tends to P and N tends to P: Thus;

$$FN \approx FP = f, \text{and } CN \approx CP = r$$
$$\Leftrightarrow CP = 2FP$$
$$\Leftrightarrow \quad \mathbf{r = 2f}$$

**Alternatively;**

Since ray AX is very close to the principle axis, then;

$$\overline{FX} \approx \overline{FP} = f \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

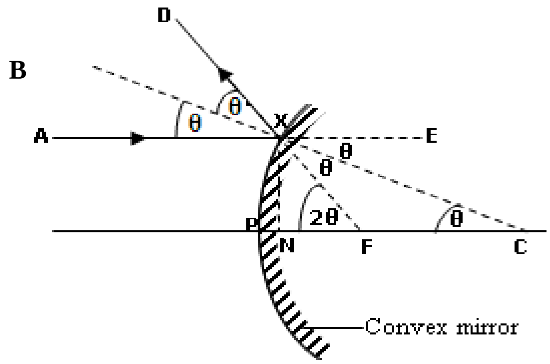
From the figure, Triangle XCF is isosceles.

$$\overline{CF} = \overline{FX} = f \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

$$\text{But } \overline{CP} = \overline{CF} + \overline{FP}$$
$$\Leftrightarrow r = f + f$$
$$\Leftrightarrow \mathbf{r = 2f}$$

**(ii) For a convex mirror**

Consider a ray parallel and near the principal axis of a diverging mirror (convex mirror), after reflection the ray appears to be coming from F.




If the normal line is drawn through X, it passes through point C.

$< AXB =< BXD = \theta$ (Laws of refllection)

$< BXD =< FXC = \theta$ . (Vertically opposite angles)

$< NFX =< FXE = 2\theta$ (Alternating angles)

For paraxial rays such as AX and small angles in radians, $\tan(XFN) =< XFN, \text{and } \tan(FXC) =< FXC$

From triangle XCN

$$\tan XCN = \tan\theta = \frac{XN}{CN} \approx \theta \ldots\ldots\ldots\ldots\ldots\ldots (i)$$

From triangle XFN

$$\tan XFN = \tan 2\theta = \frac{XN}{FN} \approx 2\theta \ldots\ldots\ldots\ldots\ldots.. (ii)$$

From equations (i) and (ii);

$$\frac{XN}{FN} = 2\left(\frac{XN}{CN}\right) \Leftrightarrow CN = 2FN$$

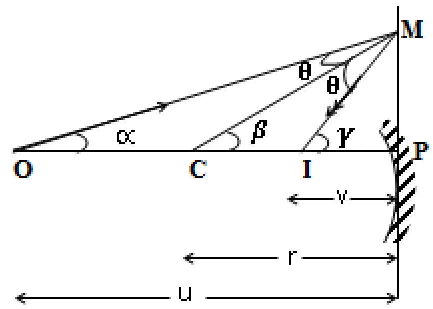
Since$<$ **XFN, and $<$ FXC are small angles in radians**, X tends to P and N tends to P: Thus;

$$FN \approx FP = f, \text{and } CN \approx CP = r$$
$$\Leftrightarrow CP = 2FP$$
$$\Leftrightarrow \quad \mathbf{r = 2f}$$

## Derivation of the Mirror Formula

**(a) Concave mirror**

- **Using a point object**




Using triangle OCM

$$\beta = \alpha + \theta$$

(Exterior angle property of a triangle).

$\theta = \beta - \alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$

Using triangle CIM

$$\gamma = \beta + \theta$$

$\theta = \gamma - \beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots\ldots\ldots. (ii)$

From equation (i) and (ii)

$$\beta - \alpha = \gamma - \beta$$

$\gamma + \alpha = 2\beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots\ldots.\ldots (iii)$

For paraxial rays and small angles in radian.

From triangle OMP:

$$\alpha \approx \tan\alpha = \frac{\overline{MP}}{\overline{OP}} = \frac{\overline{MP}}{u}$$

From triangle MCP:

$$\beta \approx \tan\beta = \frac{\overline{MP}}{\overline{CP}} = \frac{\overline{MP}}{r} \ldots\ldots (iv)$$

From triangle MIP:

$$\gamma \approx \tan\gamma = \frac{\overline{MP}}{\overline{IP}} = \frac{\overline{MP}}{v}$$

Substitute equation (iv) in equation (iii)

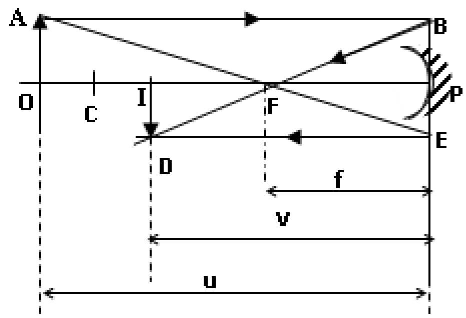
$$\gamma + \alpha = 2\beta$$

$$\left(\frac{\overline{MP}}{v}\right) + \left(\frac{\overline{MP}}{u}\right) = 2\left(\frac{\overline{MP}}{r}\right)$$

$$\overline{MP}\left(\frac{1}{v} + \frac{1}{u}\right) = \overline{MP}\left(\frac{2}{r}\right)$$

$$\frac{1}{v} + \frac{1}{u} = \frac{2}{2f} \Leftrightarrow \mathbf{\frac{1}{f} = \frac{1}{u} + \frac{1}{\mathit{v}}}$$

- **Using a finite object**




Consider a paraxial ray AB close to the principal axis.

Triangles $\triangle$ OAF and $\triangle$ FPE are similar.

$$\frac{PE}{OA} = \frac{FP}{OF} = \frac{f}{u - f}$$

$$OF = (OP - FP) = u - f$$

$$\frac{PE}{OA} = \frac{f}{u - f} \ldots\ldots\ldots\ldots\ldots (i)$$

Triangles $\triangle$ IDF and $\triangle$ FBP are similar.

$$\frac{DI}{BP} = \frac{IF}{PF} = \frac{v - f}{f}$$

$$IF = (IP - FP) = v - f$$

$$\frac{DI}{BP} = \frac{v - f}{f} \ldots\ldots\ldots\ldots (ii)$$

But : $\frac{DI}{BP} = \frac{PE}{OA}$

From equation (1) and (2)

$$\frac{f}{u - f} = \frac{v - f}{f}$$

$$f^2 = (v - f)(u - f)$$
$$f^2 = v(u - f) - f(u - f)$$
$$f^2 = uv - vf - uf + f^2$$
$$uf + vf = uv$$

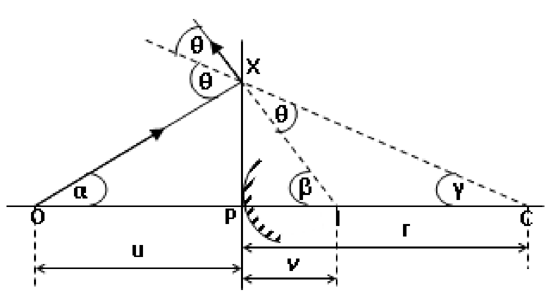
Divide all through by uvf

$$\frac{uf}{uvf} + \frac{vf}{uvf} = \frac{uv}{uvf}$$
$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

$$\frac{1}{\mathbf{f}} = \frac{1}{\mathbf{u}} + \frac{1}{\mathbf{v}}$$

**(b) Convex mirror**

(i) **<u>Using a point object:</u>**




Using triangle OXI

$$\alpha + \beta = 2\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Using triangle IXC

$$\theta + \gamma = \beta$$
$$\theta = \beta - \gamma \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Substitute Equation (ii) into equation (i)

$$\alpha + \beta = 2(\beta - \gamma)$$
$$\alpha + 2\gamma = \beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$$

For paraxial rays and angles in radian,

From triangle XOP:

$$\alpha \approx \tan\alpha = \frac{\overline{XP}}{\overline{OP}} = \frac{\overline{XP}}{u}$$

From triangle XPI:

$$\beta \approx \tan\beta = \frac{\overline{XP}}{\overline{IP}} = \frac{\overline{XP}}{v} \quad \ldots\ldots (iv)$$

From triangle MIP:

$$\gamma \approx \tan\gamma = \frac{\overline{XP}}{\overline{CP}} = \frac{\overline{XP}}{r}$$

Substitute equation (iv) in equation (iii)

$$\alpha + 2\gamma = \beta$$
$$\left(\frac{\overline{XP}}{u}\right) + 2\left(\frac{\overline{XP}}{r}\right) = \left(\frac{\overline{XP}}{v}\right)$$
$$\overline{XP}\left(\frac{1}{v} - \frac{1}{u}\right) = \overline{XP}\left(\frac{2}{r}\right)$$

$$\frac{1}{v} - \frac{1}{u} = \frac{2}{2f} \Leftrightarrow \frac{\mathbf{1}}{\mathbf{f}} = \frac{\mathbf{1}}{\mathbf{v}} - \frac{\mathbf{1}}{\mathbf{u}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (v)$$

Introducing a sign convention so that the distances are given positive or negative sign, the same equation is obtained for both concave and convex mirrors, irrespective of whether the objects and images are real or virtual.

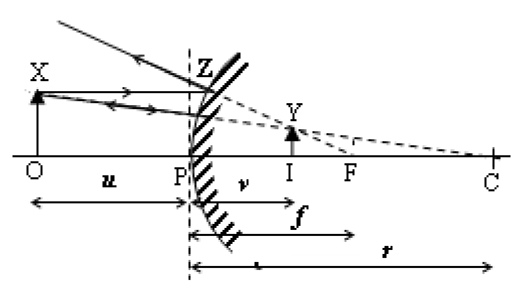
*We shall adopt the "**Real is positive**" rule which states that the real object or image distance is positive and a virtual object or image distance is negative.*

- ✓ The focal length of a concave mirror is positive and that of a convex mirror is negative.
- ✓ The radius of curvature takes the same sign as focal length.

For convex mirror, **r, f and v** are negative but **u** is positive. Thus from equation (v);

$$\frac{1}{-(f)} = \frac{1}{-(v)} - \frac{1}{u}$$
$$\frac{1}{\mathbf{f}} = \frac{1}{\mathbf{u}} + \frac{1}{\mathbf{v}}$$

(ii) **<u>Using a finite object:</u>**




Consider a paraxial ray XZ close to the principal axis.

Triangles $\triangle$ XOC and $\triangle$ IYC are similar.

$$\frac{XO}{YI} = \frac{OC}{IC} = \frac{-r + u}{-r - (-v)}$$

$IC = (PC - PI) = (-r) - (-v) = (v - r)$

$$\frac{XO}{YI} = \frac{u - r}{v - r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(i)$$

Triangles $\triangle$ IDF and $\triangle$ FBP are similar.

$$\frac{ZP}{IY} = \frac{PF}{IF} = \frac{-f}{-f - (-v)}$$

$IF = (PF - PI) = (-f) - (-v) = (v - f)$

$$\frac{ZP}{IY} = \frac{f}{f - v} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(ii)$$

$$\text{But; } XO = ZP \Leftrightarrow \frac{XO}{YI} = \frac{ZP}{IY}$$

From equation (1) and (2)

$$\frac{u - r}{v - r} = \frac{f}{f - v}$$

But; $r = 2f$

$$\frac{u - 2f}{v - 2f} = \frac{f}{f - v}$$

$$f(v - 2f) = (u - 2f)(f - v)$$

$$fv - 2f^2 = uf - uv - 2f^2 + 2fv$$

$$uv = vf + uf$$

Divide all through by **uvf**

$$\frac{uv}{uvf} = \frac{uf}{uvf} + \frac{vf}{uvf}$$

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v} \Leftrightarrow \frac{1}{\mathbf{f}} = \frac{1}{\mathbf{u}} + \frac{1}{\mathbf{v}}$$

**Question**

*Derive expression for the mirror formula using a finite object in front of a convex mirror.*

**Uses of curved mirrors**

**(a) Convex mirrors**

- The convex mirror is used as a driving mirror
- They are used in reflecting telescopes
- They are used in super markets to monitor shoppers.

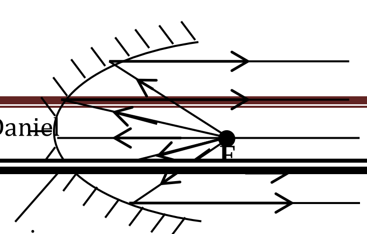
**Because; They;**

- Provides an upright image
- Have a wide field of view.

**(b) Concave mirrors**

| Concave mirror is used; | **Because; They;** |
|---|---|
| ▪ As Shaving mirrors<br>▪ As Solar concentrators.<br>▪ By dentists<br>▪ In torches and car head lamps.<br>▪ In reflecting telescopes. | -Provides a magnified upright image<br>-Form both real and virtual images. |

**(c) Parabolic mirror used as a search mirror**




Parabolic mirror

If a small lamp is placed at the Principle Focus F, of the concave mirror, it follows from the principle of reversibility of light that rays striking the mirror around a small area about the pole are reflected parallel. But these rays from the lamp which strike the mirror at points well away from pole will be reflected in different directions because a wide parallel beam is not brought to a focus at F.

The beam of light reflected from the mirror thus diminishes in intensity as its distance from the mirror increases and a concave spherical mirror is hence useless as a search mirror.

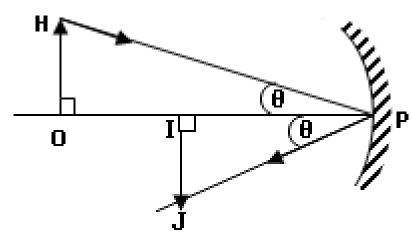
A mirror whose section is the shape of a parabola is used in search lights. Parabolic mirror has the property of reflecting a wide beam of light from a lamp at its focus as a perfectly parallel beam in which case the intensity of the reflected beam is practically undiminished as the distance from the mirror increases.

**Linear /lateral or transverse magnification**

It is the ratio of the height of image to the height of the object.

$$\textbf{Magnification; m} = \frac{\textbf{Image Height}}{\textbf{Object Height}}$$

Consider the formation of an image of finite object by spherical mirrors




Triangles $\triangle$ OHP and $\triangle$ IJP are similar.

$$\frac{IJ}{OH} = \frac{IP}{OP}$$

$$m = \frac{IP}{OP} = \frac{\text{Image distance from mirror}}{\text{Object distance from mirror}}$$

$$m = \frac{IJ}{OH} = \frac{IP}{OP}$$

$$\mathbf{m} = \frac{\textbf{Image Height}}{\textbf{Object Height}} = \frac{\textbf{Image distance}}{\textbf{Object distance}}$$

From the mirror formula;

$$\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$$

| | | |
|---|---|---|
| Multiplying through by v gives;<br>$\frac{v}{f}=\frac{v}{u}+1.$<br><br>But $\frac{v}{u}=m$<br><br>$\frac{v}{f}=m+1$<br><br>$\mathbf{m}=\frac{\mathbf{v}}{\mathbf{f}}-1$<br><br>$\mathbf{m}=\frac{\mathbf{v}-\mathbf{f}}{\mathbf{f}}$ | Multiplying through by u gives;<br>$\frac{u}{f}=1+\frac{u}{v}$<br><br>But $\frac{u}{v}=\frac{1}{m}$<br><br>$\frac{u}{f}=1+\frac{1}{m}$<br><br>$\frac{1}{\mathbf{m}}=\frac{\mathbf{u}}{\mathbf{f}}-1$<br><br>$\frac{1}{m}=\frac{u-f}{f}$<br><br>$\mathbf{m}=\frac{\mathbf{f}}{\mathbf{u}-\mathbf{f}}$ | **Note:**<br>(i) If $u=f$<br>$(object\ at\ focal\ point)$<br>$m=\frac{f}{u-f}\Leftrightarrow m=\frac{f}{f-f}$<br>$\Leftrightarrow \mathbf{m}=\infty\ (\mathbf{Infinity})$<br><br>(ii) If $v=f$<br>$(Image\ at\ focal\ point)$<br>$m=\frac{v-f}{f}\Leftrightarrow m=\frac{f-f}{f}$<br>$\Leftrightarrow \mathbf{m}=\mathbf{0}(\mathbf{Zero})$<br><br>(iii) If $u=r$<br>$\begin{pmatrix}object\ at\ centre\\ of\ cavature\end{pmatrix}$<br>$m=\frac{f}{u-f}\Leftrightarrow m=\frac{f}{2f-f}$<br>$\Leftrightarrow m=1(\text{One})$ |

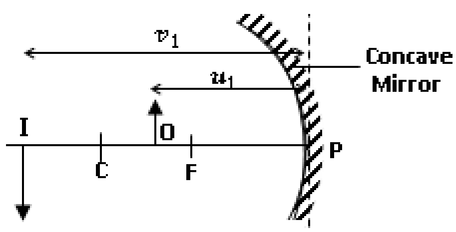
**Example: 1**

The magnification of an object in a concave mirror is $\mathbf{m}_1$. When the object is moved towards the mirror through a small distance, $\boldsymbol{d}$, the magnification becomes $\boldsymbol{m}_2$**.** Show that the focal length, ***f***, of the mirror is given by;

$$\boldsymbol{f}=\frac{\boldsymbol{d}\boldsymbol{m}_1\boldsymbol{m}_2}{\boldsymbol{m}_2-\boldsymbol{m}_1}$$

**Solution:**

**Case I:**



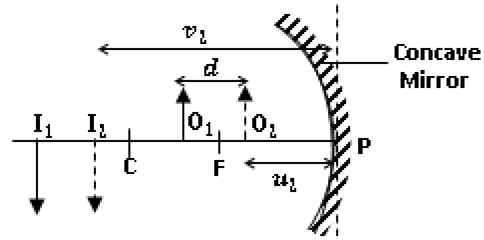

From; $m=\frac{v}{u}\Leftrightarrow m_1=\frac{v_1}{u_1}\Leftrightarrow v_1=m_1u_1$

Also; $m=\frac{v-f}{f}\Leftrightarrow m_1=\frac{m_1u_1-f}{f}$

$$m_1f=m_1u_1-f$$
$$m_1f+f=m_1u_1$$
$$f(m_1+1)=m_1u_1$$

$$f=\frac{m_1u_1}{m_1+1}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (i)$$

**Case II**




$From;m=\frac{v}{u}\Leftrightarrow m_2=\frac{v_2}{u_2}\Leftrightarrow v_2=m_2u_2$

$But\ u_2=u_1-d$

$\Leftrightarrow v_2=m_2(u_1-d)$

Also; $m=\frac{v-f}{f}\Leftrightarrow m_2=\frac{m_2(u_1-d)-f}{f}$

$$m_2f=m_2(u_1-d)-f$$
$$m_2f+f=m_2u_1-m_2d$$
$$f(m_2+1)=m_2(u_1-d)$$

$$f=\frac{m_2(u_1-d)}{m_2+1}\ \ldots\ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots\ldots\ (ii)$$

Equating equations (i) to Equation (ii) gives;

$$\frac{m_1u_1}{m_1+1}=\frac{m_2(u_1-d)}{m_2+1}$$
$$(m_1+1)[m_2(u_1-d)]=m_1u_1(m_2+1)$$
$$m_2(m_1+1)(u_1-d)=m_1u_1(m_2+1)$$
$$m_2[(m_1+1)(u_1-d)]=m_1u_1(m_2+1)$$
$$m_2[m_1u_2-m_1d+u_1-d]=m_1m_2u_1+m_1u_1$$
$$m_1m_2u_2-m_1m_2d+m_2u_1-m_2d=m_1m_2u_1+m_1u_1$$
$$m_1m_2d+m_2d=m_2u_1-m_1u_1$$
$$\boldsymbol{d}m_2(m_1+1)=u_1(m_2-m_1)$$
$$u_1=\frac{\boldsymbol{d}m_2(m_1+1)}{m_2-m_1}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (iii)$$

Put Equation (iii) into equation (i):

$$f=\frac{m_1}{(m_1+1)}\times\frac{\boldsymbol{d}m_2(m_1+1)}{m_2-m_1}$$

$$f=\frac{\boldsymbol{d}m_1m_2}{m_2-m_1}$$

**Example. 2**

A converging mirror produces an image whose length is 2.5 times that of the object. If the mirror is moved through a distance of 5cm towards the object, the image formed is 5 times as long as the object. Calculate the focal length of the mirror.

**Solution:**

From the mirror formula;

$$\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$$

Multiplying through by u gives;

$$m=\frac{f}{u-f}$$

| | |
|---|---|
| But, when $u=x$, $m=2.5$ | When $u=x-5$, $m=5$ |

| | |
|---|---|
| $2.5 = \frac{f}{x - f}$<br>$2.5(x - f) = f$<br>$2.5x - 2.5f = f$<br>$2.5x = 3.5f$<br>$x = 1.4f \ldots\ldots\ldots\ldots\ldots (i)$ | $5 = \frac{f}{x - 5 - f} \ldots\ldots\ldots\ldots (ii)$<br>$5 = \frac{f}{1.4f - 5 - f}$<br>$5 = \frac{f}{0.4f - 5}$<br>$5(0.4f - 5) = f$<br>$2f - 25 = f$<br>$f = 25 \text{ cm}$ |

**Example. 3**
A concave mirror forms a real image which is 3 times the linear size of the real object. When the object is displaced through a distance d, the real image formed is now 4 times the linear size of the object if the distance between two images position is 20cm; find.
(i). The focal length of the mirror.
(ii). The distance d.

**Solution:**
$m_1 = 3, m_2 = 4$
From definition of magnification;
$$m = \frac{v - f}{f}$$

| Case I | Case II |
|---|---|
| But, when $m_1 = 3, v_1 = x$<br>$3 = \frac{x - f}{f}$<br>$x - f = 3f$<br>$x = 4f \ldots\ldots\ldots\ldots\ldots (i)$ | But, when $m_2 = 4, v_1 = y$<br>$4 = \frac{y - f}{f}$<br>$y - f = 4f$<br>$y = 5f \ldots\ldots\ldots\ldots\ldots (ii)$ |

But the distance between the two image positions is 20 cm.
$y - x = 20$
$5f - 4f = 20$
$f = 20 \text{ cm}$

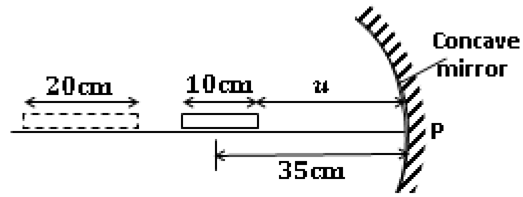
(ii)
$v_1 = x = 4f = 4(20) = 80\text{cm}$
$v_2 = y = 5f = 5(20) = 100\text{cm}$

From the mirror formula;
$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$
Case I:
$\frac{1}{20} = \frac{1}{u_1} + \frac{1}{80} \Leftrightarrow \frac{1}{u_1} = \frac{1}{20} - \frac{1}{80} \Leftrightarrow \frac{1}{u_1} = \frac{3}{80}$
$u_1 = 26.67 \text{ cm}$

Case II:
$\frac{1}{20} = \frac{1}{u_2} + \frac{1}{100} \Leftrightarrow \frac{1}{u_2} = \frac{1}{20} - \frac{1}{100} \Leftrightarrow \frac{1}{u_2} = \frac{1}{25}$
$u_2 = 25 \text{ cm}$
$d = u_1 - u_2$
$d = 26.67 - 25$
$d = 1.67 \text{ cm}$

**Example. 4:**
A rod which is 10cm long is placed along the principle axis of a concave mirror such that the mid-point of the rod is 35cm from the pole of the mirror. Calculate the radius of curvature of the mirror if it forms a real image of the rod which is20cm long.

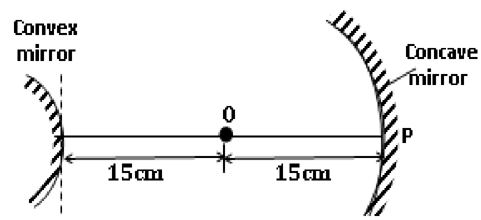
**Solution:**




Magnification, $m = \frac{v}{u} = \frac{20}{10} = 2$
$Also; m = \frac{f}{u - f} \Leftrightarrow 2 = \frac{f}{30 - f}$
$2(30 - f) = f$
$f = 20cm$
$But, r = 2f = 2(20) = 40cm$
Thus the radius of cuurvature of the mirror is 40cm

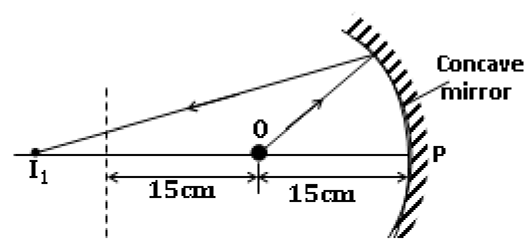
**Example. 5:**
A concave mirror of radius of curvature 25cm faces a convex mirror of radius of curvature 20cm and is 30cm from it. If an object is placed midway between the mirrors, find the nature, position of the image formed by reflection first at the concave mirror and then at the convex mirror.

**Solution:**




Concave mirror; $r_1 = 25cm$
Convex mirror; $r_2 = 20cm$

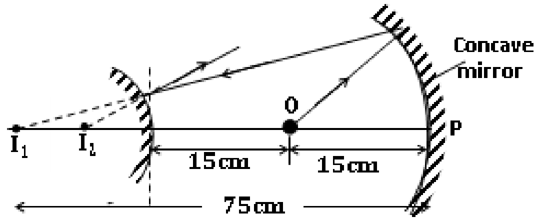
Case I: Action of the concave mirror;




$$f_1 = \frac{r_1}{2} = \frac{25}{2} = 12.5\text{cm}$$
$$u_1 = 15\text{cm}$$
From the mirror formula; $\frac{1}{f_1} = \frac{1}{u_1} + \frac{1}{v_1}$
$\frac{1}{12.5} = \frac{1}{15} + \frac{1}{v_1} \Leftrightarrow \frac{1}{v_1} = \left(\frac{1}{12.5} - \frac{1}{15}\right) = \frac{1}{75}$

$v_1 = 75\text{cm}$

Case II: Action of the convex mirror;
The convex mirror has a virtual focus.
$\mathbf{I_1}$ acts as a virtual object to the convex mirror.




$f_2 = \frac{r_2}{2} = \frac{-20}{2}$
$f_2 = -10\text{cm}$ (Because of the virtual focus)

$u_2 = -(v_1 - d)$
$u_2 = -(75 - 30) = -45\text{cm}$
$u_2 = -45\text{cm}$ (Because of the virtual object)

From the mirror formula;

$$\frac{1}{f_2} = \frac{1}{u_2} + \frac{1}{v_2}$$

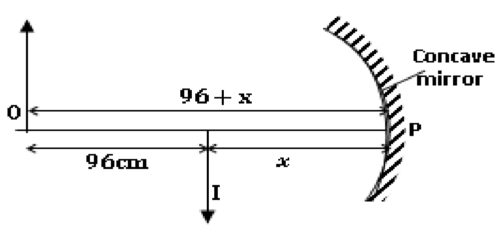
$$\frac{1}{-10} = \frac{1}{-45} + \frac{1}{v_2} \Leftrightarrow \frac{1}{v_2} = \frac{1}{-10} + \frac{1}{45}$$

$$v_2 = -12.875\text{cm}$$

Therefore, the final image of the convex mirror is vitual and 12.875 cm behind the convex mirror.

**Example. 6:**
An object is 4.0cm high. It is desired to form a real image 2.0cm tall and 96.0cm from the object. Determine the type of curved mirror required and the focal length of the mirror.

**Solution:**




$\text{Magnification}, m = \frac{h_I}{h_O} = \frac{2.0}{4.0} = 0.5 \ldots\ldots\ldots.(i)$
Also;
$\text{Magnification}, m = \frac{v}{u} = \frac{x}{96+x} \ldots\ldots..\ldots\ldots(ii)$
Equating (i) and (ii) gives;

$$0.5 = \frac{x}{96+x}$$

$$0.5(96+x) = x$$

$$x = 96\text{cm}$$

$$\Leftrightarrow u = 96 + x = 96 + 96 = 192\text{cm}$$

$$\Leftrightarrow v = 96\text{cm}$$

From the mirror formula;

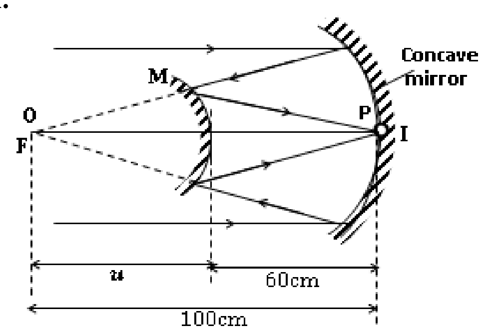
$$\frac{1}{\text{f}} = \frac{1}{\text{u}} + \frac{1}{\text{v}}$$

$$\frac{1}{\text{f}} = \frac{1}{192} + \frac{1}{96} \Leftrightarrow \frac{1}{\text{f}} = \frac{1}{64} \Leftrightarrow f = 64\text{cm}$$

Thus the cuved mirror required is concave of focal length 64 cm

**Example. 7:**
A small convex mirror is placed 60 cm from the pole and on the axis of a large concave mirror, faces the concave mirror whose radius of curvature is 200cm. The position of the convex mirror is such that the real image of a distant object is formed in the plane, O a hole drilled at the pole of a concave mirror. Sketch a ray diagram to show the arrangement and calculate the radius of curvature of the convex mirror.

**Solution:**



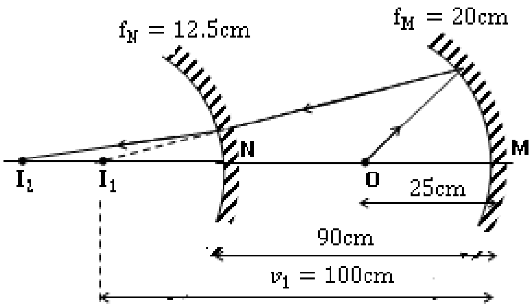

$\boldsymbol{f} = \frac{\text{r}}{2} = \frac{200}{2} = 100\text{cm}$
<u>Action of the convex mirror;</u>
$\text{u} = -(100 - 60) = -40\text{cm}$
$\text{v} = 60\text{cm}$
From the mirror formula;
$\frac{1}{\text{f}} = \frac{1}{\text{u}} + \frac{1}{\text{v}}$

$\frac{1}{\text{f}} = \frac{1}{-40} + \frac{1}{60} \Leftrightarrow \frac{1}{\text{f}} = \frac{-1}{120} \Leftrightarrow f = -120\text{cm}$
$\boldsymbol{r} = 2f = 2(-120) = -240\text{cm}$
Thus the radius of curvature of the convex mirror is 240cm

**Example. 8: UNEB 2007:**
A concave mirror, M of focal length 20.0cm is placed 90cm in front of a concave mirror, N of focal length 12.5cm. An object is placed on the common axis of M and N at apoint25.0cm in front of M. Determine the;

(i). Distance from N of the image formed by reflection, first in M and then in N.
(ii). Magnification of the image formed.

**Solution:**
**(i)**




Action of M:

$$\frac{1}{f_1} = \frac{1}{u_1} + \frac{1}{v_1}$$

$$\frac{1}{20} = \frac{1}{25} + \frac{1}{v_1}$$

$$v_1 = 100cm$$

Action of N:

$I_1$ acts as a real object for N

$u_2 = (v_1 - d)$

$u_2 = (100 - 90)$

$u_2 = 10cm$

$f_2 = 12.5cm$

$$\frac{1}{f_2} = \frac{1}{u_2} + \frac{1}{v_2}$$

$$\frac{1}{12.5} = \frac{1}{10} + \frac{1}{v_2}$$

$$v_2 = -50cm$$

Thus the final image is virtual and 50cm behind mirror N.

**(ii) Magnification, M;**

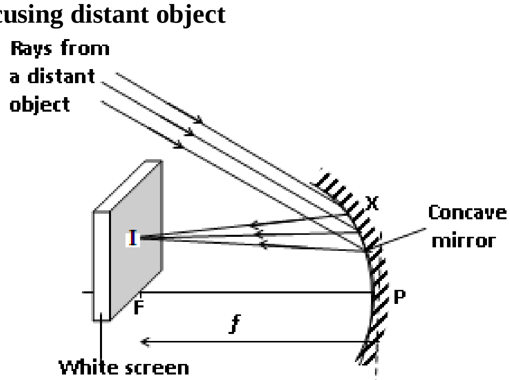
$M = m_M m_N$

$M = \left(\frac{100}{25}\right)\left(\frac{50}{10}\right) = 20$

Thus the effective magnification is 20.

**Determining the focal length of Concave mirrors**

**i) Focusing distant object**

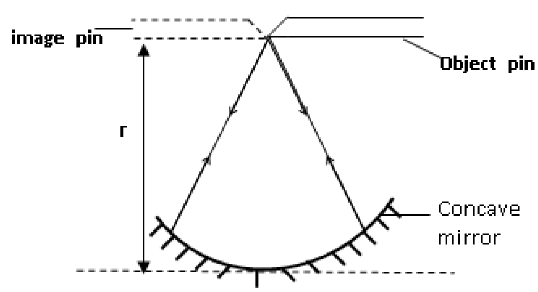



Light from a distant object such as a tree is focused on the screen.

Distance between the image (screen) and the pole of the mirror are measured using a metre- rule.

It is approximately equal to the focal length. f of the mirror.

ii) **By determining first the radius of curvature. (Self-conjugate method).**



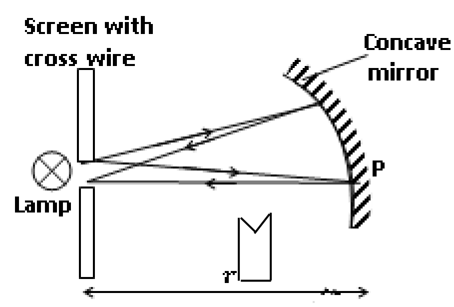

A concave mirror is placed horizontally on a bench. An optical pin is clamped horizontally on a tripod stand so that the tip lies along the principal axis of the mirror.

The position of the pin is adjusted until the position is obtained where it coincides with its image and there is no parallax between the two, i.e. there is no relative motion between the object and the image when the observer moves the head from side to side or up and down.

The distance r of the pin from the pole is measured and focal length determined,

$$f = \frac{r}{2}$$

iii) **Using an illuminated object at C**




Procedures:

The apparatus is set up as shown in the diagram.

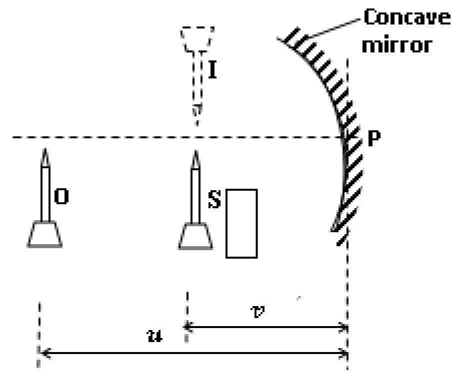
A concave mirror is moved to and fro in front of the screen until a sharp image of the cross wire is obtained on the screen.

The distance between the screen and the mirror, r is measured and recorded.

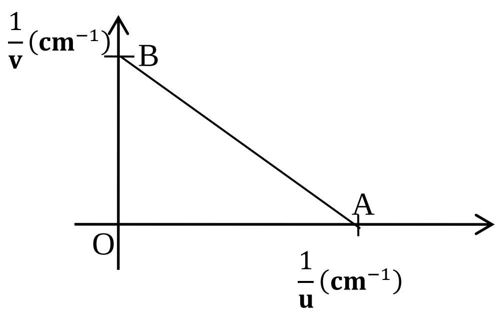
The focal length, f, of the mirror is then determined from;

$$f = \frac{r}{2}$$

iii**) Using the mirror formula.**



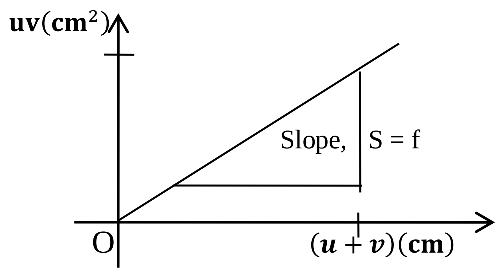
An optical pin, O, is placed in front of a concave mirror with its tip along the principle axis of the mirror.
A search pin, S, is placed between O and the mirror and its position adjusted until its tip coincides with the tip of the image, I, of the O formed by the mirror when there is no parallax between the search pin and the image, I.
The distances, v and u of the image and object respectively from the mirror are measured and recorded.
The procedure is repeated for different values of u and the corresponding values of v obtained.
The results are tabulated including values of $\frac{1}{\mathbf{u}}$ and $\frac{1}{\mathbf{v}}$
A graph of $\frac{1}{\mathbf{u}}$ against is plotted $\frac{1}{\mathbf{u}}$.




The intercepts OA and OB are read from the graph. The focal length, f of the mirror is determined from;

$$\boldsymbol{f} = \frac{1}{2}\left(\frac{1}{\mathbf{OA}} + \frac{1}{\mathbf{OB}}\right)$$

**Alternatively,**
A graph of $uv$ against $(u+v)$ may be plotted. Its slope is calculated and it is equal to the focal length of the mirror.




**<u>Theory;</u>**
From the mirror formula;

$$\frac{1}{\mathrm{f}} = \frac{1}{\mathrm{u}} + \frac{1}{\mathrm{v}} = \frac{u+v}{uv}$$

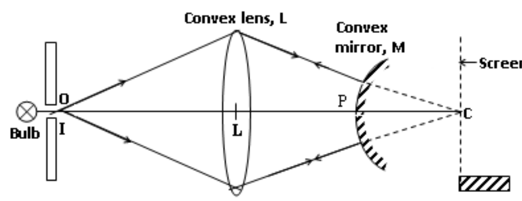
$$uv = (u+v)\mathrm{f}$$

Comparing this equation with; $y = mx + c$

$$\Leftrightarrow m = f, \quad \text{and} \quad c = 0$$

**<u>Determine the focal length of the convex mirror</u>**

**(a) Using a convex lens**

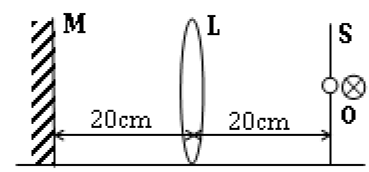



The apparatus is arranged as shown above.
An object, O is placed in front of a convex lens, L and its real image I is formed at C.
The distance LC is measured and recorded
The convex mirror whose focal length, f is required is placed between L and C with its reflecting surface facing the lens.
The lens is then moved along the axis OC until a convergent beam incident normally on the mirror forms its image at O.
The distance LP is measured

$$\mathrm{r} = \mathrm{PC} = (\mathrm{LC} - \mathrm{LP}):$$

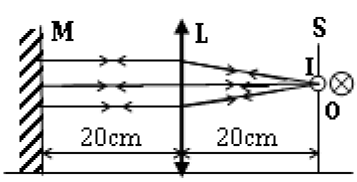
Thus f can be determined from; $\mathbf{f} = \frac{1}{2}(\mathbf{PC})$

**Example.:**
A lens, L, a plane mirror, M and a screen, S are arranged as shown in the figure below so that a sharp image of a luminous object O is formed on the screen. When the plane mirror is replaced by a convex mirror, the less has to be moved 5cm further away from the screen so as to obtain a sharp image on the screen.



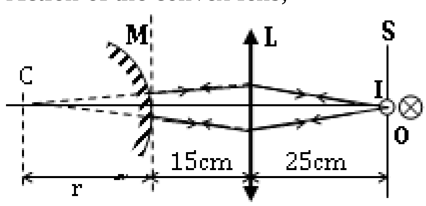

(i) Illustrate the two situations by sketch ray diagram.
(ii) Calculate the focal length of the convex mirror.

**Solution:**
Action of the plane mirror;




focal length of the lens, $f = 20$ cm

Action of the convex lens;




$\boldsymbol{u} = 25\text{cm}, \quad$ and $\boldsymbol{v} = \mathrm{r} + 15, \;\; \mathrm{f} = 20$

From the lens formula;

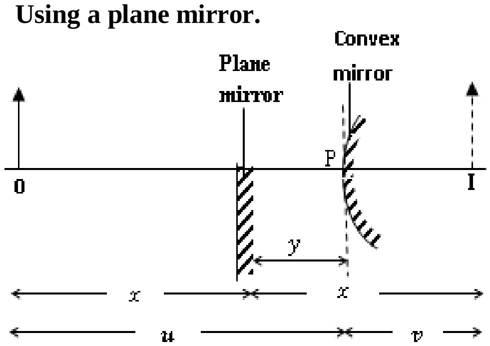
$$\frac{1}{20} = \frac{1}{25} + \frac{1}{\mathrm{r}+15}$$

$$\frac{1}{20}-\frac{1}{25}=\frac{1}{r+15} \Leftrightarrow \frac{1}{r+15}=\frac{1}{100}$$
$$r+15=100$$
$$r=85\text{cm}$$

$\boldsymbol{But}; f=\frac{r}{2}=\frac{85}{2}=42.5\text{cm}$

Thus the focal length of the convex mirror is 42.5cm

**(b) Using a plane mirror.**

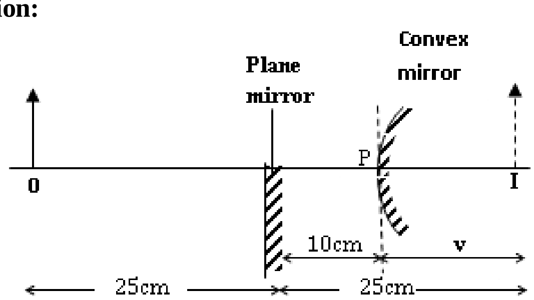



- ✓ An object O is placed in front of a convex mirror as shown in the diagram above.
- ✓ A virtual diminished image, I of the object pin is formed at point I.
- ✓ A plane mirror, M is then placed between O and P is such that it covers half the field of view of the convex mirror.
- ✓ Mirror, M is adjusted until its own image of O coincides with I by no parallax method.
- ✓ The distances x and y are measured and recorded. The distances u and v are calculated as follows;
  $$u=(x+y) \quad \text{and} \quad v=-(x-y)$$
- ✓ The focal length, f, of the convex mirror is then calculated from; $\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$

**Example.:**

A plane mirror is placed 10cm in front of a convex mirror so that it covers about half of the mirror surface. A pin 25cm in front of the plane mirror gives an image in it which is coincides with that in the convex mirror. Find the focal length of the convex mirror.

**Solution:**




Action of the convex mirror;

$u=(x+y) \quad$ and $\quad v=-(x-y)$

$u=(25+10) \quad$ and $\quad v=-(25-10)$

$u=35\text{ cm} \quad$ and $\quad v=-15\text{cm}$

From the mirror formula;

$$\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$$

$$\frac{1}{f}=\frac{1}{35}+\frac{1}{-15} \Leftrightarrow \frac{1}{f}=\frac{-4}{105} \Leftrightarrow f=-26.25\text{cm}$$

Thus the focal length of the convex mirror is 26.25 cm

**Exercise**

1. A real image is formed 40 cm from a spherical mirror, the image being twice the size of the object. What kind of mirror is it and what is the radius of the curvature?

2. An object is 4cm high. It's desired to form a real image 2cm high and 96cm from the object. Determine the type of mirror required and focal length of the mirror.

3. A dentist holds a concave mirror of focal length 4cm at a distance of 1.5 cm from the tooth. Find the position and magnification of the image which will be formed.

4. An object is placed u cm from a concave mirror and it forms an image which is 3 times its object. The object is moved by 2cm and the image formed is 5 times the size of the object. Find the:
   (i). Focal length of the mirror (f=15cm)
   (ii). Object distance(u=30cm)

5. A convex mirror whose radius of curvature is 30cm forms an image of a real object which has been placed 20cm from the mirror. Calculate the position of the image and its magnification. (v=8.6cm, and M= -0.43)

6. A convex mirror of radius of curvature of 40cm forms an image which is half the height of the object. Find the object and image position. (u=60cm).

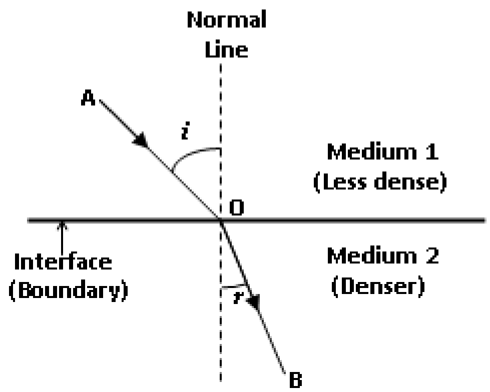
## REFRACTION OF LIGHT

Refraction is the change in speed of propagation of light due to change in optical density.

When light propagating in free space is incident in medium, the electrons and protons interact with the electric and magnetic fields of the light wave. This result in the slowing down of a light wave.

**Law of refraction**

When light passes from one medium to another, say from air glass part of it is reflected back into the previous medium and the rest passes through the second medium with its direction of travel changed.




$AO =$ **Incident ray**

$i =$ ***Angle of incidence***

$OB =$ **Refracted ray**

$r =$ **Angle of refraction**

Generally, if light is incident from a less dense medium, to a more optically dense medium, its speed reduces and it is refracted towards the normal at the point of incidence.
However, if light travels from a denser to a less dense medium, its speed increases and it is refracted away from the normal.

**Laws of Refraction**

**Law 1**. The incident ray, refracted ray and the normal at point of incidence all lie on the Same plane

**Law 2.** For any two particular media, the ratio of the sine of angle of incidence to sine of angle of refraction is constant.

i.e. $\frac{\sin i}{\sin r} = \boldsymbol{a\ constant}(\cap)$

The constant ratio $\frac{\sin i}{\sin r}$ is called the refractive index for light passing from the first to second medium.

$$\textbf{Hence; } {}_{1}\cap_{2} = \frac{\sin i}{\sin r}$$

If medium 1 is a vacuum/air, we refer the ratio as the **absolute refractive index of medium 2**, denoted by $\mathbf{n_2}$.

$$\text{Hence; } {}_{1}\cap_{2} = \frac{v_1}{v_2} = \frac{\text{Speed of light in medium 1.}}{\text{Speed of light in medium 2}}$$

If medium 1 is a vacuum/air, then;

$$\cap_2 = \frac{C}{v_2} = \frac{\text{Speed of light in vacuum.}}{\text{Speed of light in medium 2}}$$

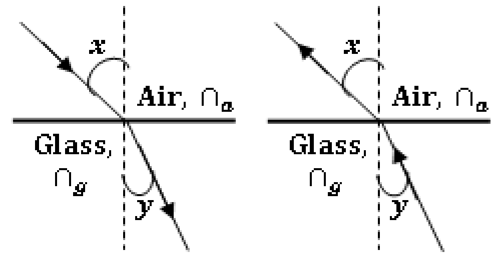
$Where, \boldsymbol{C} = 3.0 \times 10^{8}\boldsymbol{ms^{-1}}$

**Note:** For practical purposes, $\cap_{\mathbf{Vacuum}} = \cap_{\mathbf{Air}} = 1$

**Principle of Reversibility of light**

It states that if a light ray (path) after suffering a number of refractions is reversed at any stage, it travels back to the source along the same path with the same refraction.

**Relationships Refractive indices.**




The refractive index for a ray of light travelling from air (less dense medium) to glass (a denser medium) is given by;

$${}_{a}\cap_{g} = \frac{\sin x}{\sin y} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (i)$$

Applying the principle of reversibility of light to the ray, the refractive index for light travelling from glass to air is given by;

$${}_{g}\cap_{a} = \frac{\sin y}{\sin x} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (ii)$$

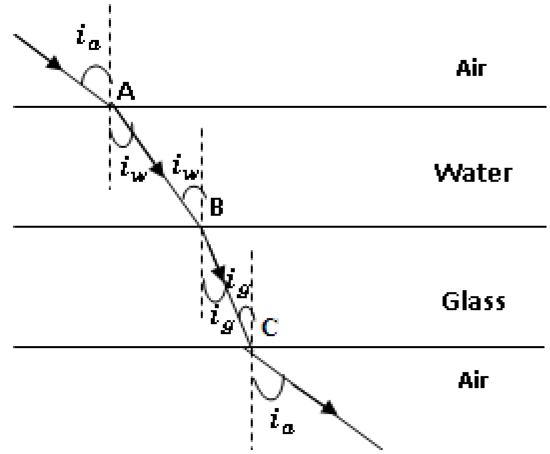
Equation (i) x Equation (ii) gives

$${}_{a}\cap_{g} \times {}_{g}\cap_{a} = \frac{\sin x}{\sin y} \times \frac{\sin y}{\sin x}$$

$${}_{\boldsymbol{a}}\cap_{\boldsymbol{g}} \times {}_{\boldsymbol{g}}\cap_{\boldsymbol{a}} = 1$$

$${}_{\boldsymbol{g}}\cap_{\boldsymbol{a}} = \frac{1}{{}_{\boldsymbol{a}}\cap_{\boldsymbol{g}}}$$

**Relationship between $\sin \boldsymbol{i}$ and $\cap$**




At A; ${}_{a}\cap_{w} = \frac{\sin i_a}{\sin i_w}$

$$\frac{\cap_{\boldsymbol{w}}}{\cap_{\boldsymbol{a}}} = \frac{\sin i_a}{\sin i_w}$$

$$\cap_a \sin i_a = \cap_w \sin i_w \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (i)$$

At B; ${}_{w}\cap_{g} = \frac{\sin i_w}{\sin g}$

$$\frac{\cap_{\boldsymbol{g}}}{\cap_{\boldsymbol{w}}} = \frac{\sin i_w}{\sin i_g}$$

$$\cap_w \sin i_w = \cap_g \sin i_g \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (ii)$$

At C; ${}_{g}\cap_{a} = \frac{\sin i_g}{\sin i_a}$

$$\frac{\cap_a}{\cap_g} = \frac{\sin i_g}{\sin i_a}$$

$\cap_a \sin i_a = \cap_g \sin i_g$ ……………………………………… $(iii)$

From equation (i), (ii) and (iii);

$\cap_a \sin i_a = \cap_w \sin i_w = \cap_g \sin i_g$

In general;

$$\cap \, \boldsymbol{sin\, i = constant}$$

Where $i$ is the angle in the medium with absolute refractive index, ∩.

**Alternatively;**

At A; $\quad {}_a\cap_w = \frac{\sin i_a}{\sin i_w}$

$\sin i_a = {}_a\cap_w \sin i_w$ …………………………………… $(i)$

At B; $\quad {}_w\cap_g = \frac{\sin i_w}{\sin g}$ ………………………………… $(ii)$

At C; $\quad {}_g\cap_a = \frac{\sin i_g}{\sin i_a}$

$\sin i_g = {}_g\cap_a \sin i_a \Leftrightarrow \sin i_a = \frac{\sin i_g}{{}_g\cap_a}$

$\sin i_a = \sin i_g \times \frac{1}{{}_g\cap_a} = \sin i_g \times {}_a\cap_g$

$\sin i_a = {}_a\cap_g \sin i_g$ ……………………………………… $(iii)$

From equation (i) and (ii)

$1 \times \boldsymbol{\sin i_a} = {}_{\boldsymbol{a}}\cap_{\boldsymbol{w}} \boldsymbol{\sin i_w} = {}_{\boldsymbol{a}}\cap_{\boldsymbol{g}} \boldsymbol{\sin i_g}$

Considering absolute refractive indices of air, water and glass

$$\boxed{\cap_a \sin i_a = \cap_w \sin i_w = \cap_g \sin i_g}$$

In general;

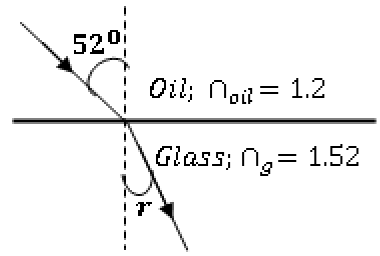
$$\cap \, \boldsymbol{sin\, i = constant}$$

Where $\boldsymbol{i}$ is the angle in the medium with absolute refractive index **∩.**

**Examples 1:**

If the angle of incidence in oil is$52^0$, find the angle of refraction in glass for a ray of light travelling from oil to glass. ($\cap_{oil} = 1.2$ and $\cap_{glass} = 1.52$)

**Solution:**




Using Snell's law;

$\cap \sin i = \text{constant}$

$\cap_{oil} \sin i_{oil} = \cap_{glass} \sin i_{glass}$

$1.2 \sin 52 = 1.52 \sin r$

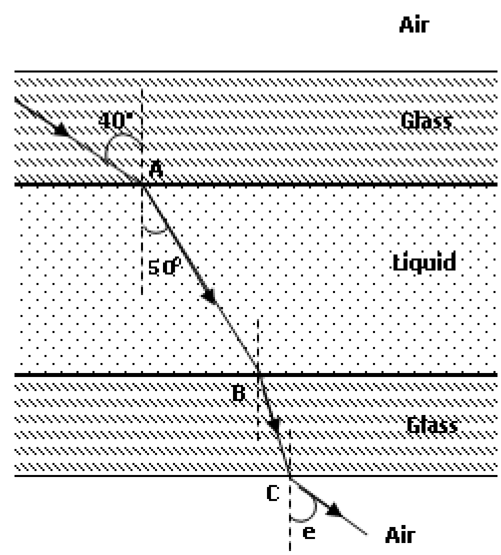
$\sin r = \left(\frac{0.9456}{1.52}\right)$

$r = \sin^{-1}\left(\frac{0.9456}{1.52}\right)$

$r = 38.47^0$

**Examples 2:**

The diagram bellow shows a liquid sandwiched between two glass slabs of refractive index 1.5. A ray of light begins from the upper glass slab and it latter emerges into air.




Find the;

(i) Refractive index of the liquid.[$n_L = 1.26$]

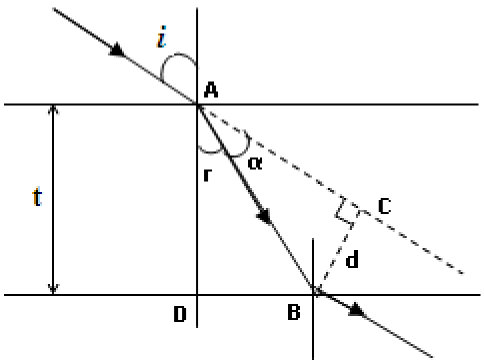
(ii) Angle of emergency in air. [$e = 74.6^0$]

**Examples 3:**

Given that in the block of refractive index n, the angles of incidence and refraction are $\boldsymbol{i}$ and $\boldsymbol{r}$ respectively. Show that the displacement BC of the incident ray is given by;

$$BC = t \sin i \left[1 - \frac{\cos i}{\sqrt{(n^2 - \sin^2 i)}}\right]$$

**Solution;**




Applying Snell's law at A:
$\cap_a \sin i_a = \cap_g \sin i_g$
$1.\sin i = \cap \sin r$

$$\sin r = \frac{sini}{\cap} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

$r + \alpha = i$ (verticall opposite angles)
$r + \alpha = i$

$$\alpha = i - r$$

From triangle ADB
$\cos r = \frac{\mathrm{AD}}{\mathrm{AB}} = \frac{\mathrm{t}}{\mathrm{AB}}$

$AB\cos r = t \Leftrightarrow \mathrm{AB} = \frac{t}{\cos r} \ldots\ldots\ldots\ldots\ldots\ldots (ii)$

From triangle ABC
$\sin\alpha = \frac{\mathrm{BC}}{\mathrm{AB}} = \frac{d}{AB}$

$d = AB\sin\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$

$$d = \frac{t}{\cos r}\sin(i - r)$$

$$d = \frac{t}{\cos r}[\sin i\cos r - \sin r\cos i]$$

$$d = \frac{t}{\cos r}\left[\sin i\cos r - \left(\frac{sini}{\cap}\right)\cos i\right]$$

$$d = t\left[\frac{\cap\sin i\cos r - \sin i\cos i}{\cap\cos r}\right]$$

$$d = t\sin i\left[\frac{\cap\cos r - \cos i}{\cap\cos r}\right]$$

$$d = t\sin i\left[1 - \frac{\cos i}{\cap\cos r}\right]$$

$$d = t\sin i\left[1 - \frac{\cos i}{\cap.\sqrt{(1 - \sin^2 r)}}\right]$$

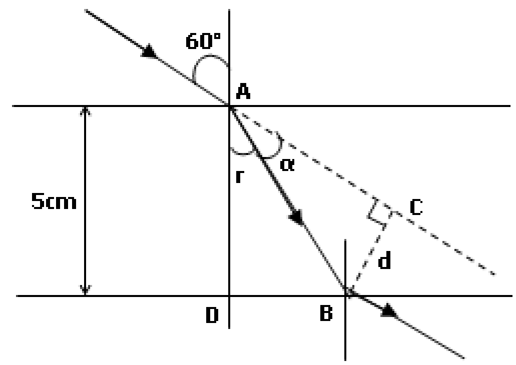
$$d = t\sin i\left[1 - \frac{\cos i}{\cap.\sqrt{1 - \left(\frac{\sin i}{\cap}\right)^2}}\right]$$

$$d = t\sin i\left[1 - \frac{\cos i}{\cap.\sqrt{\frac{\cap^2 \quad -\sin^2 i}{\cap^2}}}\right]$$

$$d = t\sin i\left[1 - \frac{\cos i}{\sqrt{\cap^2 \quad - \sin^2 i}}\right]$$

**Examples 4:**
A ray of light is incident at angle of 60° on one surface of the glass plate 5cm thick and of refractive index 1.5. Find the transverse displacement between the incident and emergent rays.

**Solution**




Applying Snell's law at A:

$$\cap_a \sin i_a = \cap_g \sin i_g$$
$$1.\sin 60 = 1.5\sin r$$
$$r = 35.26^0$$

$r + \alpha = 60^0$ (verticall opposite angles)

$$35.26^0 + \alpha = 60^0$$
$$\alpha = 24.74^0$$

From triangle ADB
$\cos r = \frac{\mathrm{AD}}{\mathrm{AB}} = \frac{5}{\mathrm{AB}}$

$\cos 35.26 = \frac{5}{\mathrm{AB}} \Leftrightarrow AB\cos 35.26 = 5 \Leftrightarrow \mathrm{AB} = \frac{5}{\cos 35.26}$

$\mathrm{AB} = 6..1\ \mathrm{cm}$

From triangle ABC
$\sin\alpha = \frac{\mathrm{BC}}{\mathrm{AB}}$

$\sin 24.74 = \frac{d}{6.1}$

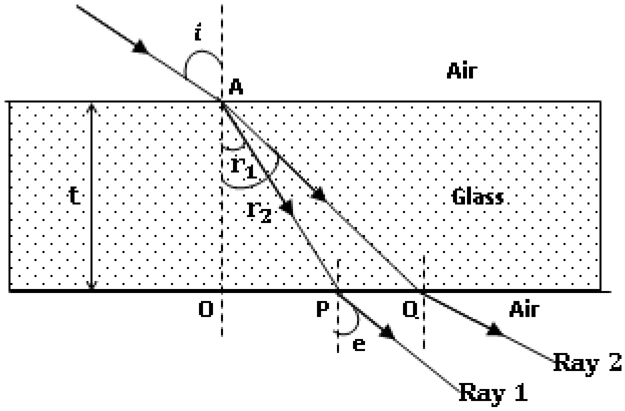
$d = 6.1\sin 24.74$
$\mathrm{d} = 2.55\mathrm{cm}$

**Examples 5:**
1. Light consisting of two colours is incident from air on glass block. The speeds of ray 1 and ray 2 are $v_1$ and $v_2$ respectively. Show that the distance PQ given by;

$$PQ = \frac{t}{c}\sin i\left(\frac{v_2}{\cos r_2} - \frac{v_1}{\cos r_1}\right)$$

Where c is speed of light in air.

**Solution**




| **For ray 1;** | **For ray 2;** |
|---|---|
| $\cap_1 = \frac{C}{v_1}$<br>Applying Snell's law at A;<br>$\cap_a \sin i = \cap_1 \sin r_1$<br>$\sin i = \cap_1 \sin r_1$<br><br>From triangle AOP;<br>$\tan r_1 = \frac{\overrightarrow{OP}}{t} \Leftrightarrow \overrightarrow{OP} = t\ \tan r_1$ | $\cap_2 = \frac{C}{v_2}$<br>Applying Snell's law at A<br>$\cap_a \sin i_a = \cap_2 \sin r_2$<br>$\sin i = \cap_2 \sin r_2$<br><br>$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$<br>From triangle AOP;<br>$\tan r_2 = \frac{\overrightarrow{OQ}}{t} \Leftrightarrow \overrightarrow{OQ} = t\ tan\ r_2$ |

Then;

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$\overrightarrow{PQ} = t\ \tan r_2 - t\ \tan r_1$$
$$PQ = t(\tan r_2 - \tan r_1)$$

$$PQ = t\left(\frac{\sin r_2}{\cos r_2} - \frac{\sin r_2}{\cos r_2}\right) = t\left[\frac{\left(\frac{\sin i}{\cap_2}\right)}{\cos r_2} - \frac{\left(\frac{\sin i}{\cap_1}\right)}{\cos r_1}\right]$$

$$PQ = t\left[\frac{\sin i}{\cap_2 \cos r_2} - \frac{\sin i}{\cap_1 \cos r_1}\right]$$

$$PQ = t\ \sin i\left[\frac{1}{\cap_2 \cos r_2} - \frac{1}{\cap_1 \cos r_1}\right]$$

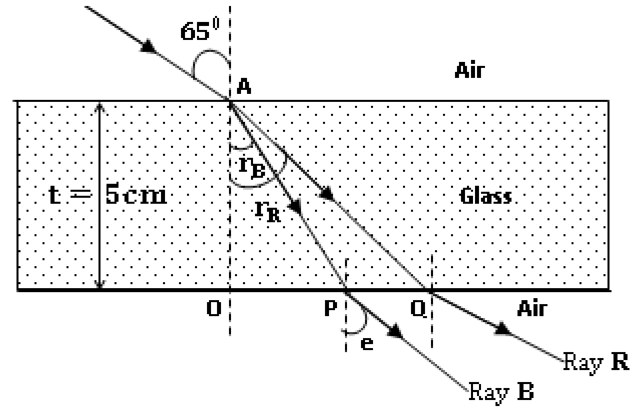
$$PQ = t\ \sin i\left[\frac{1}{\left(\frac{c}{v_2}\right)\cos r_2} - \frac{1}{\left(\frac{c}{v_1}\right)\cos r_1}\right]$$

$$PQ = t\ \sin i\left[\frac{v_2}{C\cos r_2} - \frac{v_1}{C\cos r_1}\right]$$

$$PQ = \frac{t}{C}\ \sin i\left[\frac{v_2}{\cos r_2} - \frac{v_1}{\cos r_1}\right]$$

**Example 6:**

A light ray consisting of blue and red light is incident from air to glass block. The two colours emerge from the glass block into air at point O and P respectively as shown below;




The speeds of blue and red light respectively in glass are;- $1.88 \times 10^8 \text{ms}^{-1}$ and $1.94 \times 10^8 \text{ms}^{-1}$. Find distance OP.

From definition of absolute refractive index;

$\cap = \frac{C}{v}$

For blue light;

$$\cap_B = \frac{C}{v_1} = \frac{3.0 \times 10^8}{1.88 \times 10^8} = 1.5957$$

| | |
|---|---|
| Applying Snell's law at A;<br>$\cap_a \sin i_a = \cap_B \sin r_B$<br>$\sin 65 = 1.5957 \sin r_B$<br>$\sin 65 = 1.5957 \sin r_B$<br>$\sin r_B = \frac{\sin 65}{1.5957}$<br>$r_B = 34.61^0$<br><br>For red light;<br>$\cap_R = \frac{C}{v_2} = \frac{3.0 \times 10^8}{1.94 \times 10^8}$<br>$= 1.5464$ | Applying Snell's law at A<br>$\cap_a \sin i_a = \cap_R \sin r_R$<br>$\sin 65 = 1.5464 \sin r_R$<br>$\sin 65 = 1.5464 \sin r_R$<br>$\sin r_R = \frac{\sin 65}{1.5464}$<br>$r_R = 35.88^0$<br><br>$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$<br>From triangle AOP |

$$\tan r_R = \frac{\overrightarrow{OQ}}{t} \Leftrightarrow \tan 35.88 = \frac{\overrightarrow{OQ}}{5} \Leftrightarrow \overrightarrow{OQ} = 5tan35.88$$
$$= 3.6167cm$$

From triangle AOP;

$$\tan r_B = \frac{\overrightarrow{OP}}{t} \Leftrightarrow \tan 34.61 = \frac{\overrightarrow{OP}}{5} \Leftrightarrow \overrightarrow{OP} = 5tan34.61$$
$$= 3.4506cm$$

Then;

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$\overrightarrow{PQ} = 3.6167 - 3.4506$$
$$\overrightarrow{PQ} = 0.1661\text{cm}$$

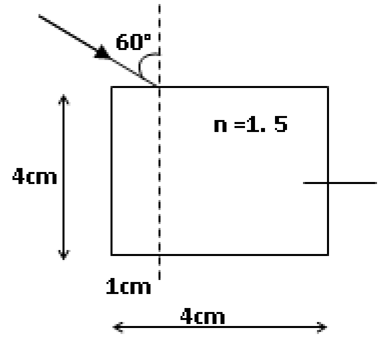
**Assignment**

1. Calculate the horizontal displacement of a ray of light incident at an angle of $57^0$ on a glass block 6cm thick and whose refractive index is 1.5. [d = 4.03cm]

2. Monochromatic light is incident on a block of transparent material placed in a vacuum. The light is refracted through an angle$\boldsymbol{\theta}$ if the block has refractive index ∩, and is of thickness $\mathbf{t}$, show that the light takes a time $T = \frac{\cap t sec\theta}{c}$ to emerge from the block where c is the speed of light in a vacuum. **(UNEB 2002)**

2. A beam of light is incident on a surface of water at an angle of 30° with the normal to the surface. The angle of refraction in water is 22°.Find the speed of light in water if it is 3 x $10^8$ $ms^{-1}$.

3. The figure below shows a glass cube of refractive index 1.5. Find the path of light




4. A glass block of refractive index $\boldsymbol{n_g}$is immersed in a liquid of refractive index $\boldsymbol{n_L}$. A ray of light travelling from the liquid to glass is incident at an angle of $\boldsymbol{\alpha}$ and is partially refracted and reflected at the interface such that the angle between the refracted and reflected rays is $90^0$.
   (i). Show that: $\boldsymbol{n_g = n_L \tan \propto}$.
   (ii). When the above procedure is repeated with the liquid removed, the angle of incidence increases by $8^0$.Given that $n_L = 1.33$, find $n_g$ and the angle of incidence at the liquid-glass interface.

$n_g = 1.397, and\ \alpha = 46.4^0$ or $n_g = 0.953, and\ \alpha = 35.8^0$

5. A fish is 3m below the surface of a pond and 2.5m from the bank. A man 2m tall stands 4m from the edge of the pond. Assuming that the sides of the pond are vertical, calculate the distance the man should move towards the edge of the pond so that he is just seen by the fish.
   $(Take, \cap_w \quad = 1.33)$: [Ans: 3.25m]

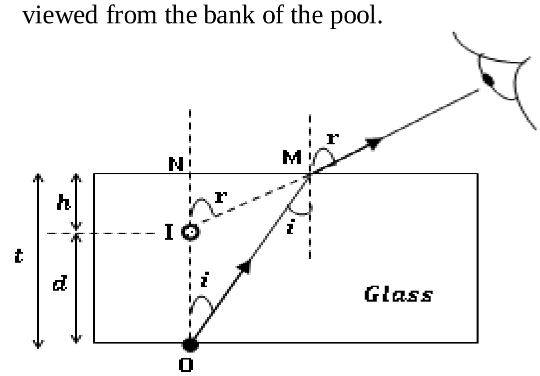
$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

From triangle AOP

## REAL AND APPARENT DEPTH

Consider a glass of thickness, t placed on top of a point object, O. A ray travelling from O is refracted at N away from the normal into the observer's eye, such that the image of O appears at I.

This effect explains why;
(i). A straw placed in a glass of water appears bent when viewed through the glass containing water.
(ii). The bottom of a pool of water appears curved and shallower near the edges than at the centre, when viewed from the bank of the pool.




Rays from O are bent away from the normal at the glass air boundary could appear to come from I, the image of O.

Using triangle OMN; $\sin i = \frac{NM}{OM}$……………………….(i)

Using triangle IMN; $\sin r = \frac{NM}{IM}$……………………….(ii)

Applying Snell's law at M

$$\cap_g \sin i_g = \cap_a \sin r$$
$$\cap \sin i \ = \sin r$$
$$\cap = \frac{\sin r}{\sin i} = \frac{\left(\frac{NM}{IM}\right)}{\left(\frac{NM}{OM}\right)} = \frac{NM}{IM} \times \frac{OM}{NM} = \frac{OM}{IM}$$
$$\cap = \frac{OM}{IM} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (iii)$$

Viewing the object O directly above it, angle $\boldsymbol{i} = 0^0$
$OM \approx ON$, and $IM \approx IN$. Thus from equation (ii);

$$\cap = \frac{OM}{IM} = \frac{ON}{IN} = \frac{t}{h}$$
$$\cap = \frac{\mathbf{t}}{\mathbf{h}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (iv)$$
$$\cap = \frac{\textbf{Real depth}}{\textbf{Apparent depth}}$$

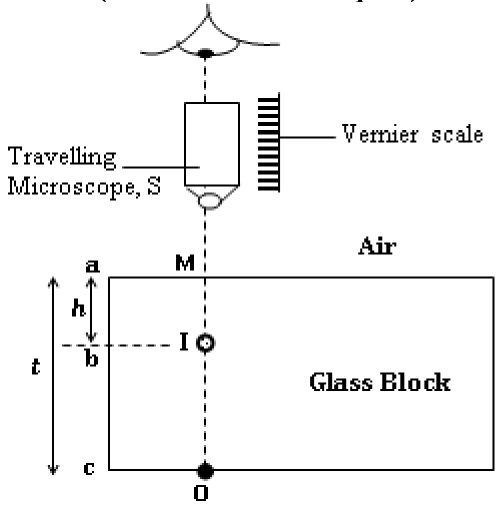
$\textbf{Displacement}, \boldsymbol{d} = \textbf{Real depth} - \textbf{Apparent depth}$

$$d = t - h;$$

From equation (iv), $h = \frac{t}{\cap}$

$$d = t - \frac{t}{\cap}$$
$$\boldsymbol{d} = \boldsymbol{t}\left(1 - \frac{1}{\cap}\right)$$

**Measurement of refractive index using real and apparent depth method (For both solids and liquids)**




A traveling microscope is focused on a pencil dot O on a sheet of white paper lying on a bench, and the reading, **c**, on the microscope scale noted.

If the refractive index of glass is required, a block of the material is placed over the dot, and the microscope refocused The microscope is raised until the particles are focused at I. I is the image of O as seen through the block, Let the reading of the microscope be, **b.**

A travelling microscope, S is focused on the top M of the glass block sprinkled with visible lycopodium powder. The microscope is raised until the powder is in focus. Note and record the reading of the scale**, a**.

$$\text{Refractive index, n} = \frac{\text{Real depth}}{\text{Apparent depth}} = \frac{\text{t}}{\text{h}} = \frac{|\mathbf{a}-\mathbf{c}|}{|\mathbf{a}-\mathbf{b}|}$$

**NOTE:**
For a liquid in a beaker, similar procedures are followed.

**Example:1**
1. A microscope is focused on a scratch on the bottom of the beaker. Turpentine is poured into the beaker to depth of 4cm and it is found to raise the microscope through a vertical distance of 1.28cm to bring the scratch back into focus. Find the refractive index of turpentine.

**Solution:**

$$\text{n} = \frac{\text{Real depth}}{\text{Apparent depth}} = \frac{|4|}{|4-1.28|} = \frac{4}{2.72} = 1.47$$

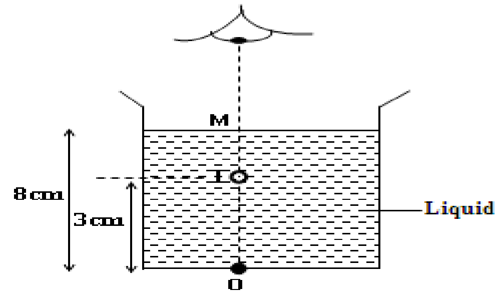
**Example:2**
2. A microscope is first focused on a scratch on the inside of the bottom of an empty glass dish. Water is then poured in and it is found that the microscope has to be raised by 1.2cm for refocusing. Chalk dust is sprinkled on the surface of water and this dust comes into focus when the microscope is raised an additional 3.5cm. Find the refractive index of water.

**Solution:**
Displacement, d = 1.2, Apparent depth =3.5,
Real depth = 1.2+ 3.5= 4.7 cm.

$$\text{n} = \frac{\text{Real depth}}{\text{Apparent depth}} = \frac{|4.7|}{|3.5|} = 1.35$$

**Example:3**
An object placed under a container full of a liquid appears to be at a point 3cm above the bottom of the container. If the height of the container is 8cm, calculate the refractive index of the liquid.

**Solution:**




$$\text{Refractive index, n} = \frac{\text{Real depth}}{\text{Apparent depth}}$$

$$\text{n} = \frac{8}{8-3} = \frac{8}{5}$$

$$\text{n} = 1.60$$

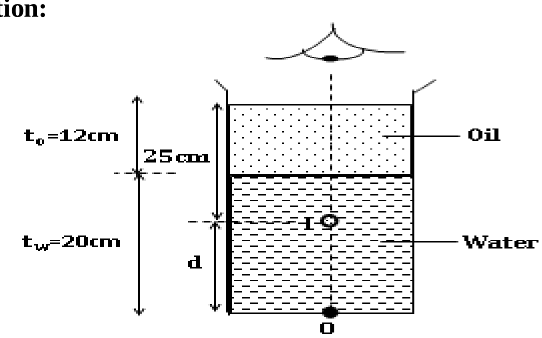
Alternatively;
Given; $\text{d} = 3\text{cm}, \text{t} = 8\text{cm}$
Then from;

$$d = t\left(1 - \frac{1}{\cap}\right)$$

$$3 = 8\left(1 - \frac{1}{\cap}\right)$$

$$\frac{8}{\cap} = 5 \Leftrightarrow 8 = 5\cap \Leftrightarrow \cap = \frac{8}{5} = 1.60$$

**Example:4**
A layer of oil 12 cm thick is poured on top of water 20cm thick in a transparent vessel. An object placed at the bottom of the vessel is viewed at a distance of 25cm from the top of the surface of oil. Determine the refractive index of oil.
(Refractive index of water = $\frac{4}{3}$ **)**
**Solution:**



Total displacement, $d = (t_w + t_o) - h$

$$d = (12 + 20) - 25 = 7\text{cm}$$

$$Displacement\ by\ water, d_w = t_w\left(1 - \frac{1}{\cap_w}\right)$$

$$d_w = 20\left[1 - \frac{1}{(^4/_3)}\right] = 20\left(1 - \frac{3}{4}\right)$$

$$d_w = 5\text{cm}$$

$$Displacement\ by\ Oil, d_o = t_o\left(1 - \frac{1}{\cap_o}\right)$$

$$d_o = 12\left(1 - \frac{1}{\cap_o}\right)$$

But;

Total displacement, $d = d_w + d_o$

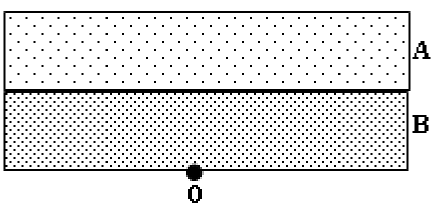
$$7 = 5 + 12\left(1 - \frac{1}{\cap_o}\right)$$

$$7 = 5 + 12 - \frac{12}{\cap_o}$$

$$\cap_o = 1.2$$

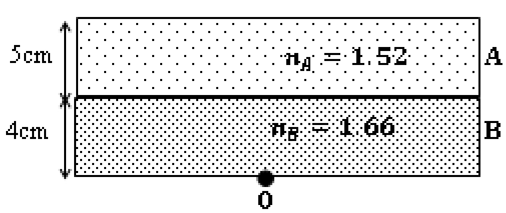
**Example:5 UNEB: 2006;**

Two parallel sided blocks A and B of thickness 4.0cm and 5.0cm respectively are arranged such that B lies on object O as shown in the diagram bellow.




Calculate the apparent displacement of O when observed from directly above. Given that the refractive indices of A and B are 1.52 and 1.66 respectively.

**Solution:**




From: Displacement, d;

$$d = t\left(1 - \frac{1}{\cap}\right)$$

For, block, A: $d_A = 4\left(1 - \frac{1}{1.52}\right) = 1.368$

For, block, B: $d_B = 5\left(1 - \frac{1}{1.66}\right) = 1.590$

Total displacement, $d = d_A + d_B$

$$d = 1.368 + 1.590$$

$$d = 2.958 \text{ cm}$$

**Exercise:**

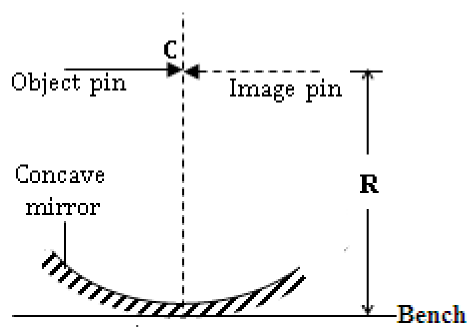
1. UNEB: 1998.

2. An object is placed20cm below the surface of water of refractive index $\frac{4}{3}$. Where would the object appear to be if the observer is inclined at a small angle of about 0.7 radians to the normal to the surface of water?
[15cm below the water surface]

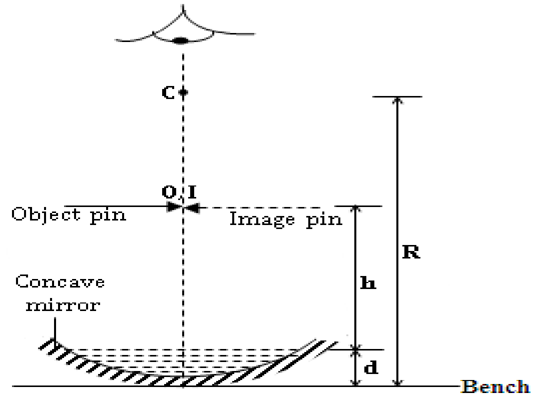
3. A tank of water contains a glass slab 8cm thick and of refractive index 1.6. Above the slab is a liquid of depth 4.5cm and refractive index 1.5. Up on this liquid floats 6cm of water of refractive index $\frac{4}{3}$. To an observer looking down from above, determine the apparent position of a mark at the bottom of the tank.
[d = 6cm from the bottom of the tank] or
[h = 12.5cm from the top of the water surface]

4. A tank contains a slab of glass 5cm thick and refractive index 1.5. On top of the glass, there is a liquid of depth 4cm. Given that an object at the bottom of the tank is displaced by 1.8cm as seen from above, find the refractive index of the liquid. **[$\cap_L$ = 1.03].**

**Determining refractive index of a liquid using a concave mirror**




A concave mirror is placed facing up on a bench. An optical pin is clamped horizontally in a retort stand so that its apex is a long the principal axis of the mirror. The height of the pin above the mirror is adjusted until the pin coincides with its image and there is no parallax. The height R of the pin above the pole of the mirror is noted.



A little quantity of the specimen liquid whose refractive index is required is dropped into the mirror to a depth, d.
The height of the pin is adjusted until the pin again coincides with its image. The height **h** of the pin above the liquid surface is measured and recorded.
The refractive index, $\cap_L$ of the specimen liquid is determined from;

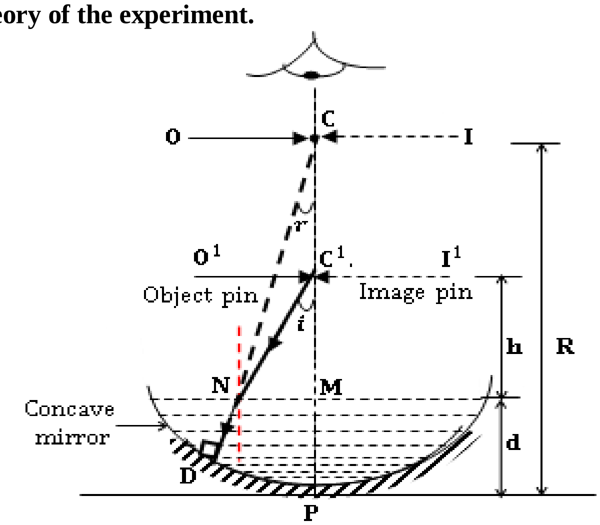
$$\cap_L = \frac{\text{Real depth}}{\text{Apparent depth}} = \frac{R-d}{h}$$

$$\cap_L = \frac{\mathbf{R-d}}{\mathbf{h}}$$

**NOTE:**
For a mirror with a large radius of curvature (large surface of the mirror) and a small quantity of a liquid in it, the refractive index of the liquid is given by;

$$\cap_L = \frac{R}{h+d}$$

**Theory of the experiment.**




From triangle, $NC^1M$;

$$\sin i = \frac{NM}{NC^1} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (i)$$

From triangle, NCM;

$$\sin r = \frac{NM}{NC} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (ii)$$

Considering Refraction at point N; Using Snell's law;

$$\cap_a \sin i_a = \cap_L \sin r_L$$

$$\sin i = \cap_L \sin r$$

$$\cap_L = \frac{\sin r}{\sin i} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (iii)$$

Putting equations (i) and (ii) into equation (iii) gives;

$$\cap_L = \frac{\sin r}{\sin i} = \frac{NM}{NC^1} \div \frac{NM}{NC} = \frac{NM}{NC^1} \times \frac{NC}{NM} = \frac{NC}{NC^1}$$

$$\cap_L = \frac{NC}{NC^1}$$

But NC and $NC^1$ are paraxial rays, thus they are close to the principle axis.

$$NC \approx MC, \text{and } NC^1 \approx MC^1 \Leftrightarrow \frac{NC}{NC^1} \approx \frac{MC}{MC^1}$$

Therefore; $\cap_L = \frac{MC}{MC^1} \Leftrightarrow \cap_L = \frac{R-d}{h}$

For a large mirror surface and a small quantity of liquid in it, M is very close to P. Hence;

$$MC \approx PC, \text{and } MC^1 \approx PC^1 \Leftrightarrow \frac{MC}{MC^1} \approx \frac{PC}{PC^1}$$

Therefore; $\cap_L = \frac{PC}{PC^1} \Leftrightarrow \cap_L = \frac{R}{h+d}$
Where d is the depth of the liquid in the concave mirror.

**Examples:**
1. A concave mirror of radius of curvature 40.0cm contains a liquid to a height of 2.0cm. A pin clamped horizontally and viewed from above is observed to coincide with its image when it is 27.0cm above the surface of the liquid.

**Solution:**

$$d = 2.0\text{cm}, MC^1 = h = 27.0\text{cm}, R = 40.0\text{cm}, \cap_L = ?$$

$$\cap_L = \frac{R-d}{h} = \frac{40.0-2.0}{27.0} = \frac{38}{27} = 1.41$$

2. A liquid placed on a concave mirror to a depth of 2cm. An object held above the liquid coincides with the image when it is 5cm from the pole of the mirror. If the radius of curvature of mirror is 6cm. calculate the refractive index of the liquid.

**Solution:**

$$d = 2\text{cm}, PC^1 = H = 5\text{cm}, R = 6\text{cm}, \cap_L = ?$$

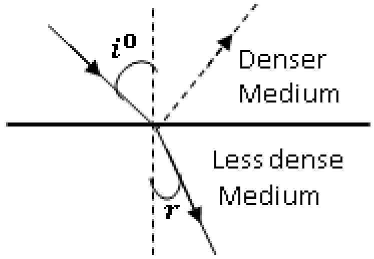
$$h = H - d = 5 - 2 = 3$$

$$\cap_L = \frac{R-d}{h} = \frac{6-2}{5-2} = \frac{4}{3} = 1.33$$

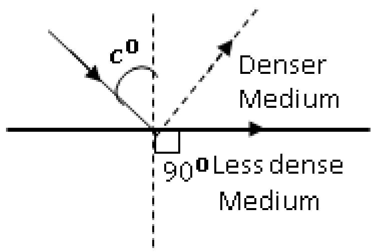
3. UNEB 2005.
4. UNEB 2015 No.2 (e).

## TOTAL INTERNAL REFLECTION AND CRITICAL ANGLE

Consider monochromatic light propagating from a dense medium and incident on a plane boundary with less dense medium at a small angle of incidence. Light is partly reflected and partly refracted.



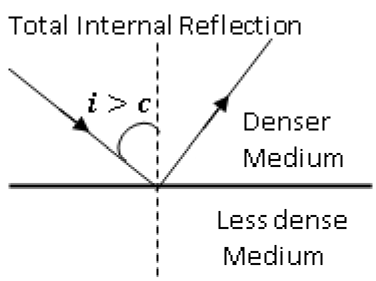

As the angle of incidence is increased gradually, a stage is reached when the refracted ray grazes the boundary between the two media




The angle of incidence **c** is called the critical angle.

Hence **critical angle** is the angle of incidence in a denser medium which makes the angle of refraction in a less dense medium 90°.

When the angle of incidence is increased beyond the critical angle, the light is totally internally reflected in the denser medium. Total internal reflection is said to have occurred.



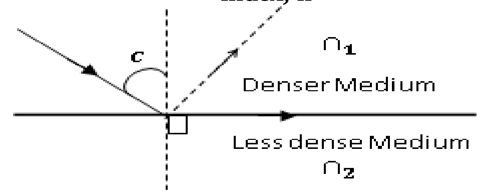

Hence **Total Internal Reflection** is the process where all the incident light energy is reflected back in the optically denser medium when the critical angle is exceeded.

**Conditions for Total internal reflection to occur.**

(i). Light must be moving from an optically denser medium to a less dense medium.

(ii). The angle of incidence in the optically denser medium must exceed (greater than) the critical angle. **[$i > c$ ].**

**Relationship between critical angle, c, and refractive index, n**




Using Snell's law:

$\cap_1 \sin i_1 = \cap_2 \sin r_2$

$\cap_1 \sin c = \cap_2 \sin 90$

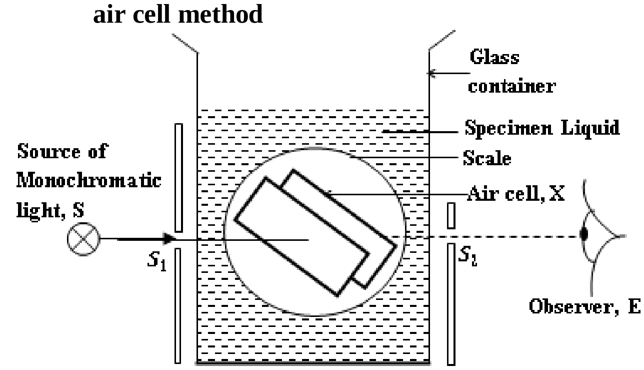
$$\sin c = \frac{\cap_2}{\cap_1}$$

If the lens dense medium is air or a vacuum;

$$\mathbf{\sin c} = \frac{1}{\cap_1}$$

**Applications of Total Internal Reflection**

**a) Determining refractive index of a liquid using the air cell method**

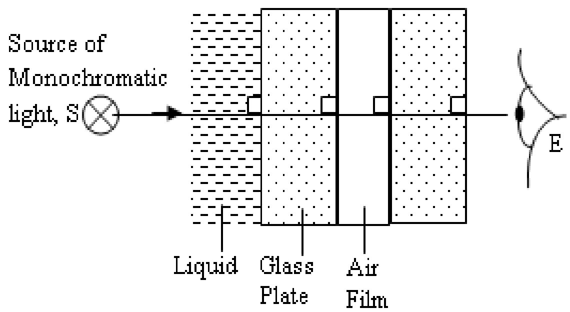



- ✓ Two thin plane parallel glass plates such as microscope slides are cemented together containing a thin film of constant thickness, hence forming an air cell.
- ✓ The liquid whose refractive index is required is poured inside a glass vessel and the air cell X is placed in the specimen liquid.
- ✓ Monochromatic light is directed normally onto the air cell, X and is observed from the opposite side at E.
- ✓ The air cell is rotated until light is suddenly cut off from E. The revolution of the pointer on the circular scale $\boldsymbol{\theta}_1$ is read and noted.
- ✓ The air cell is then rotated in the opposite direction until light is again suddenly cut off from E. The reading on the circular scale $\boldsymbol{\theta}_2$ is read and noted.
- ✓ The angle Θ between the two positions is measured. The refractive index, $\cap_L$ of the liquid is calculated from;

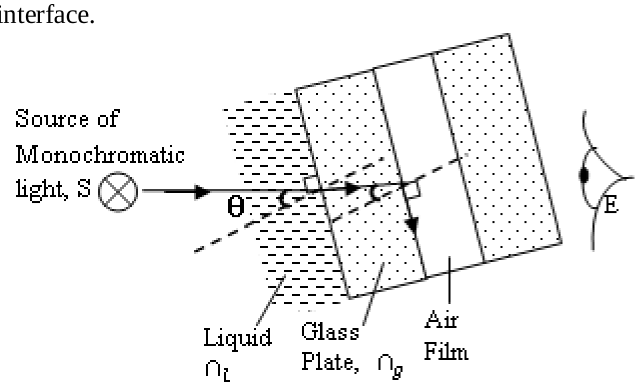
$$\cap_L = \frac{1}{\sin\theta} \text{ : Where, } \theta = \frac{\theta_1+\theta_2}{2}.$$

**Theory of the Experiment**

When light is incident normally on the air cell, the light ray from S passes directly to E.




When the air cell is slightly rotated along a vertical axis through an angle θ, such that light from the liquid is incident on the glass plate at an angle, θ, light will just be cut off from E, only when the refracted light just grazes the glass - air film interface.




At the water - Glass interface;

$$\cap_L \sin\theta = \cap_g \sin r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

At the Glass - air interface;

$$\cap_g \sin r = \sin 90 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

But from the diagram, r = c. Thus from equations (i) and (ii);

$\cap_L \sin\theta = \cap_g \sin c = \sin 90$

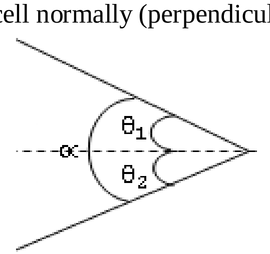
$$\cap_L \sin\theta = \sin 90$$

$$\sin\theta = \frac{1}{\cap_L}$$

$$\cap_L = \frac{1}{\mathbf{\sin\theta}}$$

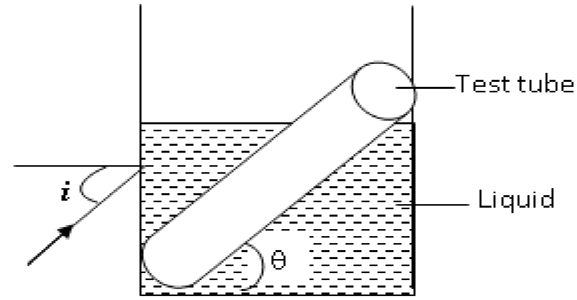
θ is the angle of incidence in the in the liquid medium. It is It is equal to the angle of rotation when the air cell is rotated from its normal position untill when light is cut off from, E.

It is better to rotate the air cell in two opposite directions because it is difficult to locate the exact initial position when light hits the air cell normally (perpendicularly).




Monochromatic light (light of only one colour) should be used because if white light is used, the colours of its spectrum will be refracted at different angles since the degree of refraction depends on the wave length of the incident light.

**Example:**
A test tube is inclined at an angle θ in a beaker containing a liquid as shown below.

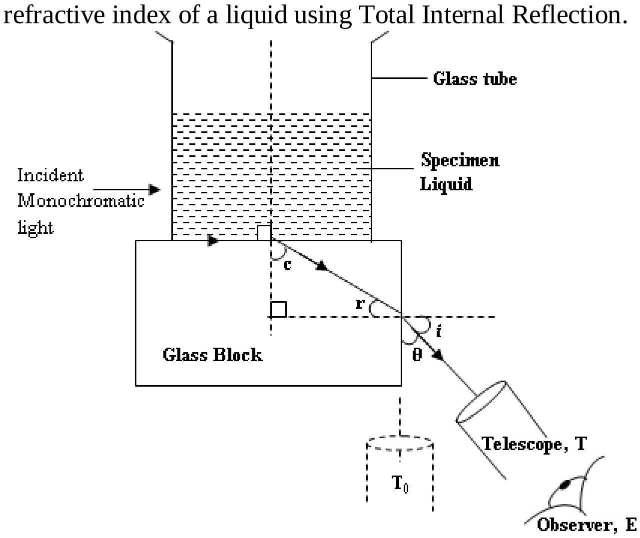



(i). Explain why the test tube appears silvery when viewed from above the liquid surface.
*Because light grazes at the surface of the test tube.*

(ii). If the incident light makes an angle of $27^0$, and the test tube is inclined at $\theta = 55^0$. Calculate the refractive index of the liquid. [$\cap_L = 1.19$]

**b) Determination of Refractive index of a liquid using the Pulfrich Refractometer.**

A refractometer is an instrument which measures the refractive index of a liquid using Total Internal Reflection.




The specimen liquid is poured in a glass tube placed on top of a glass block of known refractive index, $\cap_g$ .
Monochromatic light is directed onto the glass tube containing the liquid as shown above.
The telescope, T is adjusted so that it is in line with the side of the glass block.
The telescope, T is then turned anti- clockwise to observe light refracted through the glass block, until light is cut off from the observer at E.

The angle of rotation of the telescope, θ, between telescope positions $T_0$ and T is measured and recorded.
Then; $i = 90 - \theta$.
The refractive index of the liquid is then determined from;

$$\cap_L = \sqrt{(\cap_g{}^2 - \sin^2 i)}$$

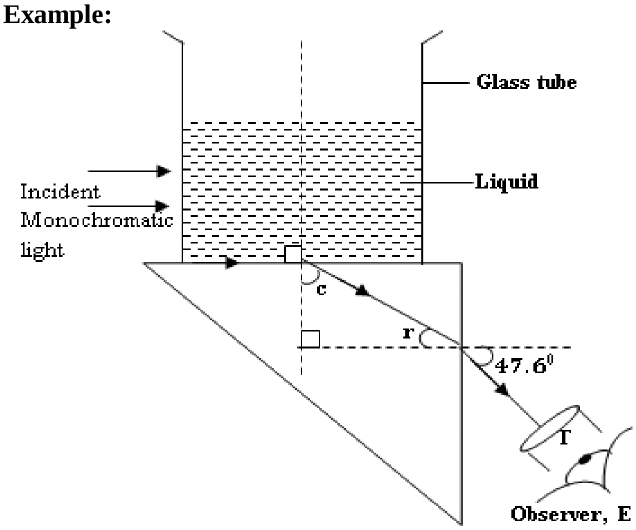
**Theory of the experiment.**
At the liquid – glass interface;
From Snell's law,
$\cap_L \sin 90 = \cap_g \sin c$
$\cap_L = \cap_g \sin c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$
At the Glass - air interface;
$\cap_g \sin r = \cap_a \sin i$
$\cap_g \sin r = \sin i$

But, $r = 90 - c$
$\cap_g \sin(90 - c) = \sin i$
$\sin i = \cap_g \cos c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$

$(\text{Equation (i)})^2 + (\text{Equation (ii)})^2$ gives;
$\cap_L{}^2 + \sin^2 i = \cap_g{}^2 \cos^2 c + \cap_g{}^2 \sin^2 c$
$\cap_L{}^2 + \sin^2 i = \cap_g{}^2 (\cos^2 c + \sin^2 c)$
$\cap_L{}^2 + \sin^2 i = \cap_g{}^2$
$\cap_L{}^2 = \cap_g{}^2 - \sin^2 i$

$$\cap_L = \sqrt{(\cap_g{}^2 - \sin^2 i)}$$

**Example:**



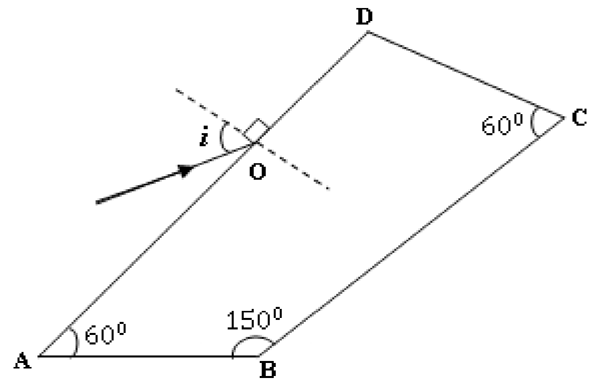

The diagram above shows a path followed by a ray of monochromatic light through a right angled prism of refractive index, 1.52. If the light emerges in air, find the refractive index of the liquid.

**Solution:**
At the liquid – glass interface;
From Snell's law,
$\cap_L \sin 90 = \cap_g \sin c$
$\cap_L \sin 90 = 1.52 \sin c$
$\cap_L = 1.52 \sin c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$

At the glass - air interface;
$\cap_g \sin r = \cap_a \sin i$
$1.52 \sin r = \sin 47.6$
$r = 29.1^0$

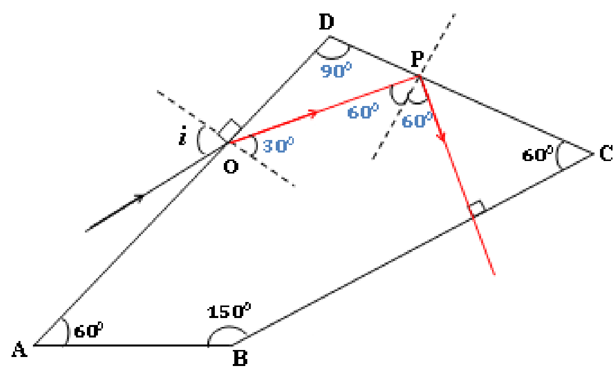
But, $r = 90 - c \Leftrightarrow c = 90 - r = 90 - 29.1 = 60.9^0$
Substituting for c into equation (i) gives;
$\cap_L = 1.52 \sin c$
$\cap_L = 1.52 \sin 60.9^0$
$\cap_L = 1.33$

**Example: UNEB 2000:**
A ray of light is incident on face AD of a glass block of refractive index 1.52 as shown in the figure below.



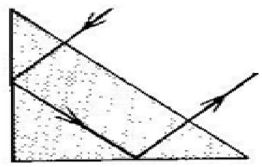
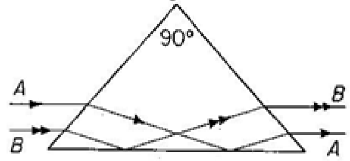

If the ray emerges normally through face BC after total internal reflection, calculate the angle of incidence, i.

**Solution:**




Total internal reflection occurs at face DC.
At P: Applying laws of reflection of light.
$i = r = 60^0$

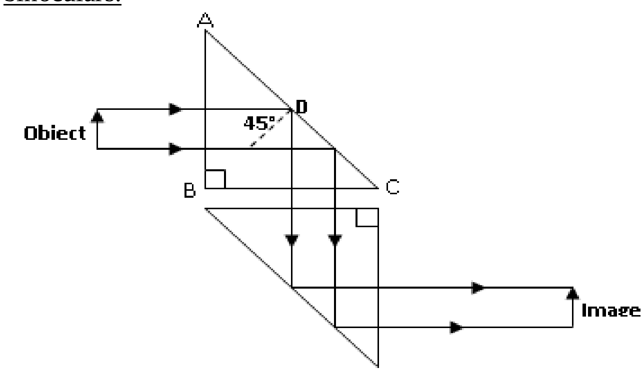
At O: Applying Snell's law.
$\cap_a \sin i = \cap_g \sin r$
$\sin i = 1.52 \sin 30$
$i = \sin^{-1}(1.52 \sin 30)$
$i = 49.46^0$

| (c) Turning a ray through $180^0$ | (d) Turning a ray through $360^0$ |
|---|---|
|  |   |

**(e) Prism periscopes**

The angle of incidence at D is equal to 45 which is greater than the critical angle hence the light is totally internally reflected at D and emerges at right angles to BC.

Two such prisms are used in submarines periscopes and prism binoculars.




**Advantages of the prism periscope over the mirror periscopes**

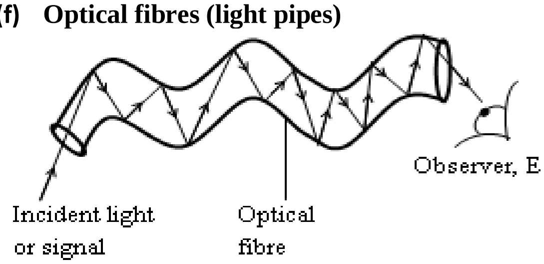
- Mirrors form multiple images due to partial reflections in the body of the mirror while in prisms there is a single reflection on the surface i.e. the prisms form sharp images.
- Some light is absorbed at the reflecting surface of the mirror therefore images are faint but there is total internal reflection at surface of the prism therefore the images are brighter.
- With time, mirrors tarnish but prism surface does not tarnish therefore a long lasting.
- The silvering in plane mirrors wears out while there is no silvering in prisms.

**Disadvantages**

The prisms are more expensive and bulkier than mirrors.

A periscope is used to observe objects over obstacles.

**(f) Optical fibres (light pipes)**




Light can be trapped by total internal reflection inside a bent glass rod and piped along a curved path.

The inner surface has slightly higher refractive index than the outer surface making it a slightly denser medium.

The beam is reflected from side practically count loss and emerges only at the end of the rod where it strikes the surface almost normally.

A single, very thin solid glass fibre behaves in the same way and if several thousands are topped together a flexible light pipe is obtained.

They are used to view inaccessible spots which cannot easily be seen. In medicine such instruments are called endoscopes.

Optical fibres can be used by doctors and engineers to light up some awkward spot for inspection.

Optical fibres are also applied in communication systems where they are used to transmit information using a modulated laser beam from one end to another as a result of repeated total internal reflection at the glass boundary, even if the fibre is bent or twisted.

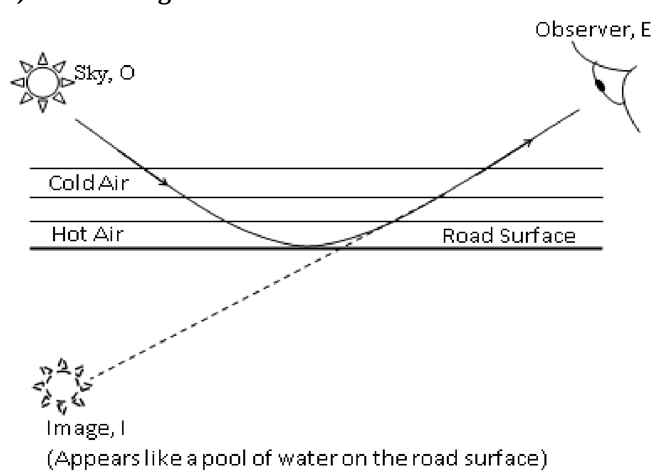
Modern telephone cables are optical fibres using laser light.

**Disadvantage**

There is leakage of light at places of contact between the glass (because light is absorbed by glass).it can be avoided by coating each fibre with glass of lower refractive index than its own hence encouraging total internal reflection.

**Effects of Total Internal Reflection**

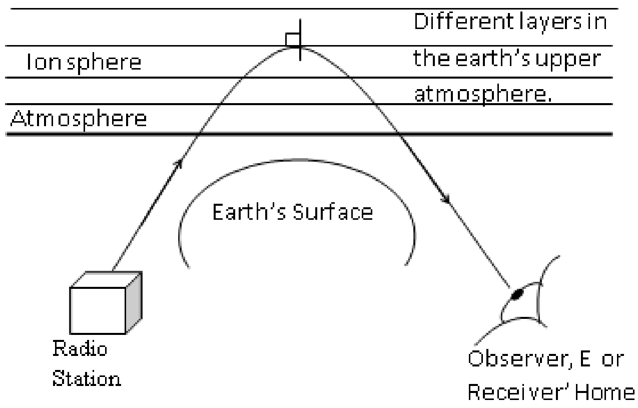
**a) Mirage**




On a hot day (clear), the air near a road gets heated and the density of air decreases as the road is approached. A ray of light from the blue sky is progressively refracted a way from the normal until the critical angle is reached when light is totally internally reflected.

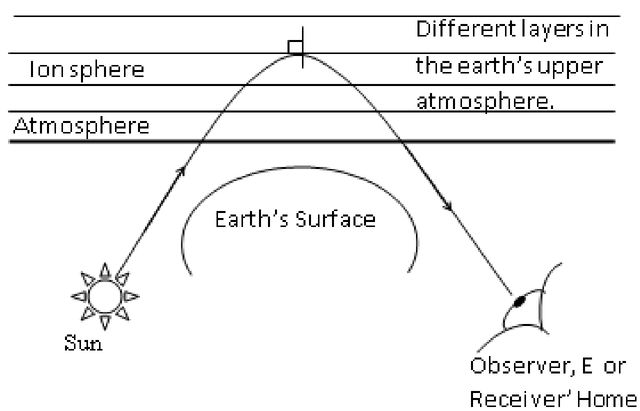
An observer at E sees an inverted image of object O as through there was a pool of water on the road due to the blue colour of the sky.

**b) Transmission of radio waves.**



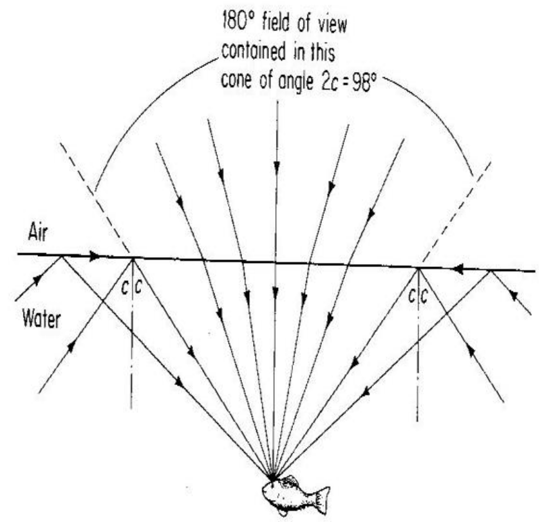

Radio waves are electromagnetic waves containing electric and magnetic fields.
Radio waves are refracted just like light waves.
Radio waves from a radio station on the earth's surface are beamed (or directed) towards the ion sphere (Region of ionized gases at very high altitudes).
As the optical density of the ion sphere decreases with the height, the waves are gradually refracted away from the normal until such a height when the waves are totally internally reflected. Hence they are received by a receiver on the other side of the earth.

**NOTE:**



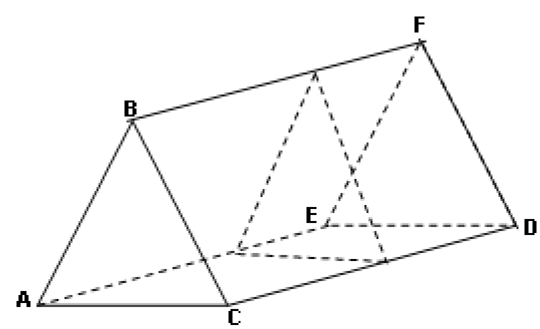

Light from the sun can be received before the sun comes to the horizon, because:
Rays of light from the sun move to the upper atmosphere and find regions (or layers) of different optical densities hence suffering refraction as they move from one region to another.
At one stage, the angle of incidence becomes greater than the critical angle and total internal reflection occurs.
The rays of light start bending down wards towards the observer at a home stead.
However, the observer should be on the opposite side of the earth where darkness would be expected. But because of total internal reflection, light from the sun can be received even before the sun comes on the horizon.

**c) Fish's eye view**



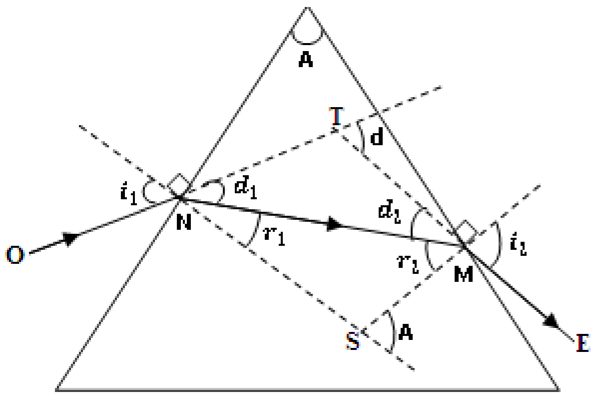

- A fish in water can have a water field of view as it can see an object normally at A
- If angle i is less than the critical angle, it can see an object B by reflection.
- It can also see an object at the bank C of the lake if the angle of incidence is equal to the critical angle.
- If *i* is greater than the critical angle an object at D can be seen by total internal reflection.

**Refraction by glass prism**




**BF**= Refracting edge
AB and BC = Refracting surface
AC and ED = Base
ABC and DEF = Principle section (or any other plane perpendicular to the refracting edge).
Angle ABC = Refracting angle or angle of the prism.

**Representation of a Prism.**




Rays: ON = Incident ray; Angle $i_1$ = angle of incidence
NM = Refracted ray; Angle $r_1$ = angle of refraction
ME = Emergent ray; Angle $i_2$ = angle of emergence
Lines: NS and MS = Normal lines on either sides.
Angle A = angle of the prism or refracting angle.

**Deviation of light by a prism**

Considering Deviation at N,

$i_1 = r_1 + d_1 \Leftrightarrow d_1 = i_1 - r_1 \ldots\ldots\ldots\ldots\ldots\ldots (i)$

Considering Deviation at M,

$i_2 = r_2 + d_2 \Leftrightarrow d_2 = i_2 - r_2 \ldots\ldots\ldots\ldots\ldots\ldots (ii)$

From Triangle NMT:

$d = d_1 + d_2 \ldots\ldots\ldots\ldots\ldots\ldots (iii)$

Putting equations (i) and (ii) into Equation (iii) gives:

$d = (i_1 - r_1) + (i_2 - r_2)$
$d = (i_1 + i_2) + (-r_1 - r_2)$
$d = (i_1 + i_2) - (r_1 + r_2) \ldots\ldots\ldots\ldots\ldots\ldots (iv)$

From Triangle NMS:

$A = r_1 + r_2 \ldots\ldots\ldots\ldots\ldots\ldots (v)$
$\boldsymbol{d = (i_1 + i_2) - A} \ldots\ldots\ldots\ldots\ldots\ldots (vi)$

**Deviation of light by a small angle prism**

Consider monochromatic light incident almost normally on a small angle prism of refractive index n placed in air. The angle of the prism is assumed to be so small that **sin A ≈ A** , with A in radians.

Applying Snell's law at N:

$\cap_a \sin i_1 = \cap \sin r_1$
$\sin i_1 = \cap \sin r_1$

But for small angles in radians, $\sin i_1 \approx i_1$ *and* $\sin r_1 \approx r_1$

$\Leftrightarrow i_1 = \cap\, r_1 \ldots\ldots\ldots\ldots\ldots\ldots (vii)$

Applying Snell's law at M:

$\cap \sin r_2 = \cap_a \sin i_2$
$\sin i_2 = \cap \sin r_2$

But for small angles in radians, $\sin i_2 \approx i_2$ *and* $\sin r_2 \approx r_2$

$\Leftrightarrow i_2 = \cap\, r_2 \ldots\ldots\ldots\ldots\ldots\ldots (viii)$

From the general expression for the deviation by any prism, i.e, from equation (vi);

$d = (i_1 + i_2) - A$
$d = (\cap\, r_1 + \cap\, r_2) - A$
$d = \cap\,(r_1 + r_2) - A$

From Triangle NMS or equation (v); $A = r_1 + r_2$

$d = \cap\,(A) - A$
$\boldsymbol{d = A(\cap - 1)}$

**Minimum Deviation, $d_{min}$ caused by a prism**

Minimum deviation is the deviation that occurs when;

- The angle of incidence is equal to the angle of emergence.

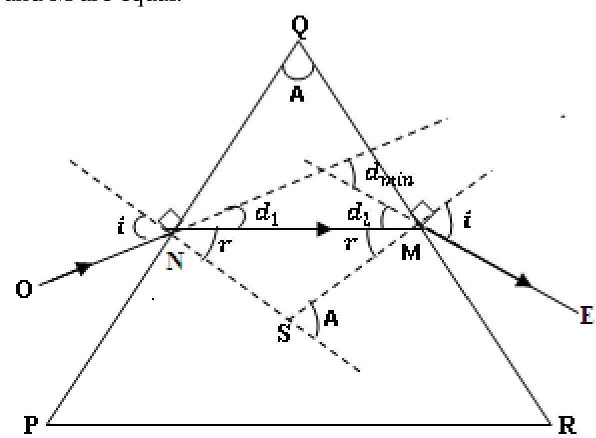
$\boldsymbol{i_1 = i_2 = i}$ and $\boldsymbol{r_1 = r_2 = r}$

- The light ray passes through the prism symmetrically.

**Note:** As the angle of incidence increases, the angle of deviation, d decreases. This continues until a certain angle of incidence, $\boldsymbol{i_{min}}$ where the angle of deviation begins to increase. The angle of deviation corresponding to the angle of incidence, $\boldsymbol{i_{min}}$ is called the angle of minimum deviation, $\boldsymbol{d_{min}}$.

**Relationship between ∩, A and $d_{min}$**

Consider a ray ON, NM and MI passing symmetrically through the prism. The angles made with the normal in air at N and M are equal.




From Triangle NMS:

$A = r + r = 2r \Leftrightarrow \boldsymbol{r = \frac{A}{2}} \ldots\ldots\ldots\ldots\ldots\ldots (\boldsymbol{i})$

$d_{min} = d_1 + d_2$
$d_{min} = (i - r) + (i - r)$
$d_{min} = (i + i) + (-r - r)$
$d_{min} = (2i) - (2r)$
$d_{min} = 2i - A$

$\boldsymbol{i = \frac{d_{min} + A}{2}} \ldots\ldots\ldots\ldots\ldots\ldots (ii)$

Applying Snell's law at N:

$\cap_a \sin i = \cap \sin r$
$\sin i = \cap \sin r$

$\cap = \frac{\sin i}{\sin r} \ldots\ldots\ldots\ldots\ldots\ldots (iii)$

Putting equations (i) and (ii) into equation (iii) gives;

$$\cap = \frac{\sin\left(\frac{d_{min}+A}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

This is the relationship between refractive index, n of the prism, refracting angle, A and the angle of minimum deviation, $d_{min}$.

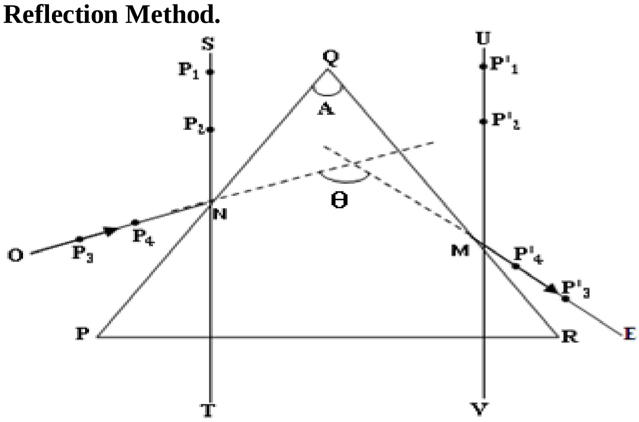
If the prism is surrounded by a medium of refractive index $\cap_m$ , then, Applying Snell's law at N, gives;

$$\cap_m \sin i = \cap \sin r$$

$$\cap_m \sin\left(\frac{d_{min}+A}{2}\right) = \cap \sin\left(\frac{A}{2}\right)$$

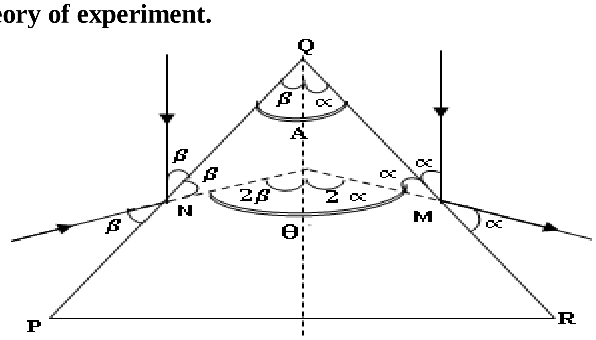
$$\cap = \frac{\cap_m \quad \sin\left(\frac{d_{min}+A}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

**Measurement of the refracting angle, A of a prism by Reflection Method.**




A white sheet of paper is stuck on a soft board using drawing pins.
Two parallel lines ST and UV are drawn on the white sheet of paper and the prism is placed with its apex as shown above.
Two optical pins $P_1$ and $P_2$ are fixed along ST and pins P3 and P4 are fixed such that they appear to be in a straight line with the images of $P_1$ and $P_2$ as seen by reflection from face PQ.
The procedure is repeated for face QR
The prism is removed and angle X measured and recorded.
The refracting angle, A is then calculated from; $\mathbf{A = \frac{\theta}{2}}$

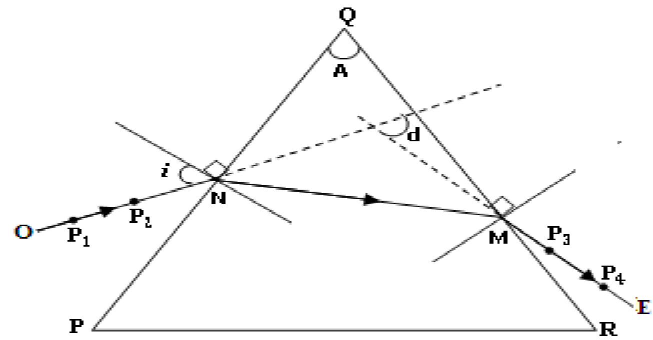
**Theory of experiment.**




$$A = \propto + \beta$$
$$\theta = 2\propto + 2\beta$$
$$\theta = 2(\propto + \beta)$$
$$\theta = 2(A)$$
$$\boldsymbol{A} = \frac{\boldsymbol{\theta}}{2}$$

**Experiment to determine the angle of minimum deviation**

**a) By no parallax method using optical pins**

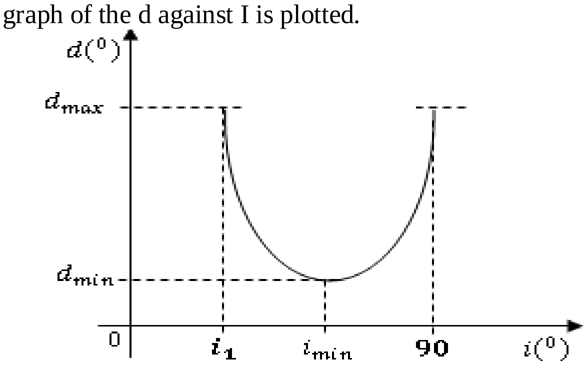



**Procedures**

Pins $P_1$ and $P_2$ are placed along line ON making an angle of incidence, i with the normal at side PQ.
Looking through side, QR, of the prism, pins $P_3$ and $P_4$ are fixed such that they appear to be in a straight line with the images of $P_1$ and $P_2$.
The angle of deviation, d is measured and recorded.
The procedure is repeated for different angles of incidence, i, and the results tabulated.

| $i(^0)$ | $d(^0)$ |
|---|---|
| - | - |
| - | - |

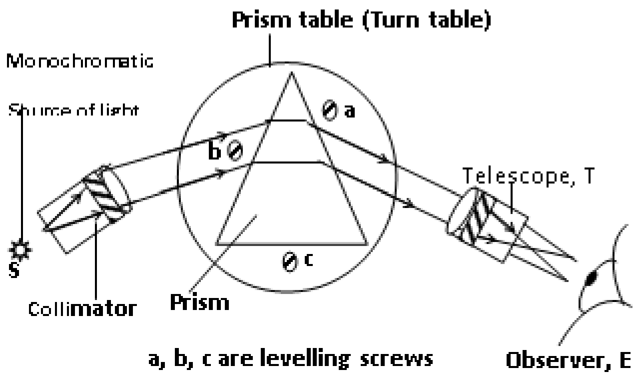
A graph of the d against I is plotted.




The minimum point of the curve, $d_{min}$ is read off and is equal to the angle of minimum deviation.

**b) Using a spectrometer**

A spectrometer is an optical instrument used to study light from different sources




The spectrometer essentially consists of the following parts;

1. Collimator: This is a tube with an adjustable slit at one end and a convex lens at the other end. The collimator is always fixed.

2. Turn-table: This is a circular metal plate with a circular scale and it rotates around the vertical axis.

3. Telescope: It consists of a set of cross wires. It is mounted horizontally and is free to rotate about the vertical axis. The lenses in the collimator and the telescope are achromatic.

**Initial adjustments of the spectrometer:**
Before the spectrometer is used, a number of adjustments are made. These include;

i) Adjustment of the Telescope:

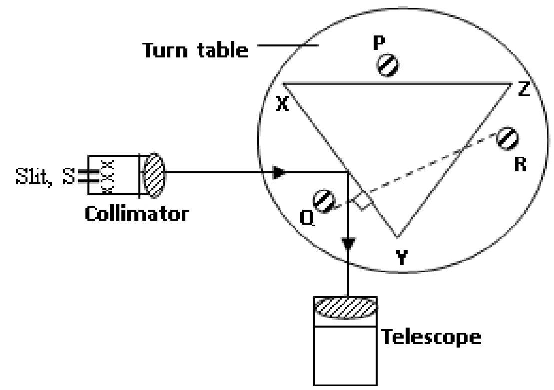
The eye piece of the telescope is adjusted so that the cross wires are clearly seen.
The telescope is then pointed at distant objects and the telescope objective is adjusted so that the image of the image of the distant object is clearly formed between the cross wires. Through this adjustment the telescope is set to receive parallel light.

ii) Adjustment of the Collimator:

The prism is removed and the collimator is moved in or out of the collimator tube until its image as seen through the telescope is in sharp focus at the cross wires.
Through this adjustment the collimator is set to produce parallel light.

iii) Levelling of the prism table:

The prism is placed on the table in such a way that one refracting face XY is perpendicular to the line joining the levelling screws Q and R.

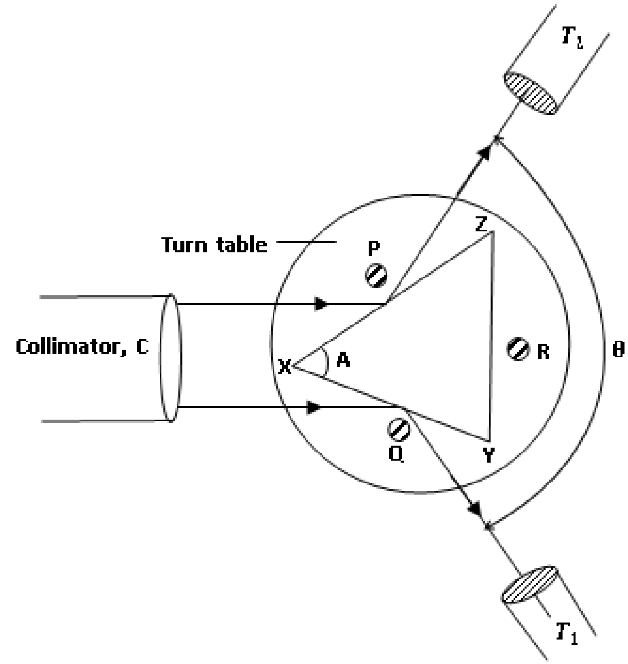



The telescope is turned through 90° and the table turned so that light reflected by XY enters the telescope. The image of the slit of the collimator formed at the cross wires is observed. If the images are not at the centre of the cross wires levelling screw Q is Adjusted until the image is in the centre of the field of view.
The table is then turned so that the refracting surface XZ reflects light into the telescope. The levelling screw P is adjusted if necessary so that the image of the Slit is in the centre of the field of view. The prism table is now level and the spectrometer is ready for use.

**Uses of a spectrometer:** A spectrometer is used in the;

(i). Measurement of the refracting angle, A of the prism.
(ii). Measurement of the angle of minimum deviation.
(iii). Measurement of the refractive index of the prism.

**(i) Measurement of the refracting angle A using prism spectrometer**

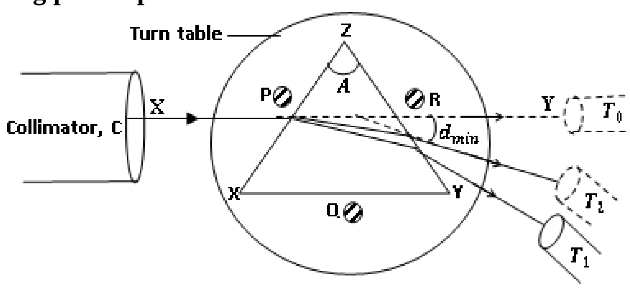



The prism table is turned so that the refracting angle of the prism faces the collimator. The telescope is turned to position $T_1$ to receive light reflected by the refracting surface XY.

Then adjust the telescope so that an image of the slit is observed at the centre of the cross wires. The reading $\theta_1$ on the circular scales of the spectrometer are noted.
The telescope is then moved to position $T_2$ to receive light reflected from the refracting surface XZ of the prism. The new reading $\theta_2$ on the circular scales of the spectrometer are noted.

The value of $\theta = (\theta_2 - \theta_1)$ is obtained and is equal to twice the refracting angle, A or $\theta = 2A \Leftrightarrow \mathbf{A} = \frac{\theta}{2}$.

**(ii) Measurement of the angle of minimum deviation $d_{min}$ using prism spectrometer.**




The Prism table is fixed and the prism is removed. The telescope is then turned $\mathbf{T_0}$ to receive un deviated light directly from the collimator. Note the telescope positions, $\theta_0$.

The prism is placed with the refracting angle pointing a way from the collimator as shown above.
The telescope is rotated to position, $T_1$ in such a direction that the image of the slit remains centered on the cross wires. Note the telescope positions, $\theta_1$.

Rotate the turn-table so that the angle of incidence is reduced and then rotate the telescope to keep the image of the slit on the cross wires.

The image and the telescope slowly approach XY. However, at $T_2$, a stage is reached where the image of the slit begins to move in the opposite direction to that in which it had been moving. Note the telescope positions, $\theta_2$.
This is the position of minimum deviation. The angle between the telescope positions $T_2$ and $T_0$, is measured from the scale of the spectrometer and noted. It is the angle of minimum deviation. $\mathbf{d_{min}}$.

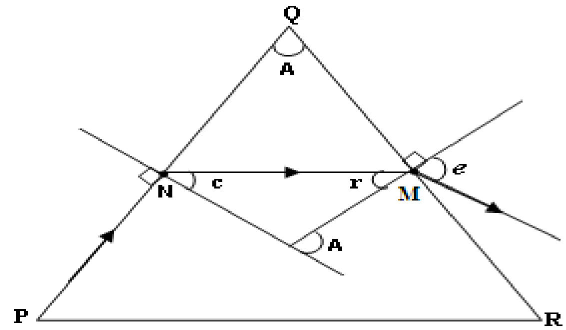
$$\mathbf{d_{min}} = (\boldsymbol{\theta}_2 - \boldsymbol{\theta}_0)$$

**(iii) Measurement of refractive index of glass in form of a prism using prism spectrometer**
This involves the Measurement of the refracting angle, A of the prism and the angle of minimum deviation, $d_{min}$.

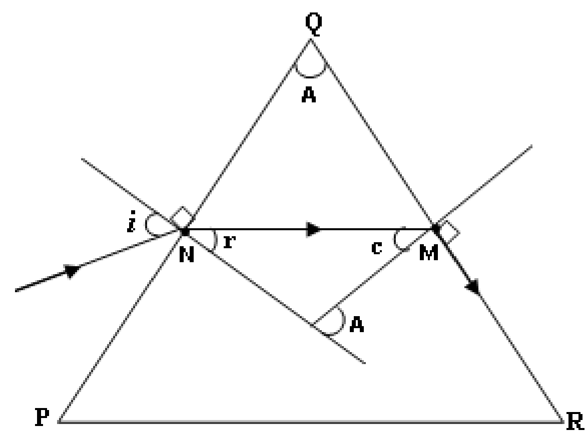
Then the refractive index, ∩ of the glass prism is calculated from;

$$\cap = \frac{\sin\left(\frac{d_{min}+A}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

**Grazing Incidence**
Grazing incidence occurs when the incident ray moving along refracting surface PQ is refracted such that it emerges out of the prism through refracting surface QR at an angle of emergence, e. The angle of incidence is $90^0$.




**Grazing Emergence**
Grazing emergence occurs when a ray incident on refracting surface PQ is refracted such that it emerges out of the prism along refracting surface QR. The angle of emergence is $90^0$.




Applying Snell's law at N:
$\cap_a \sin i = \cap \sin r$
$\sin i = \cap \sin r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$

Applying Snell's law at M:
$\cap \sin c = \cap_a \sin e$
$\cap \sin c = \sin 90$
$\cap \sin c = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$

But, also;
$\boldsymbol{r} + \boldsymbol{c} = \boldsymbol{A} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$

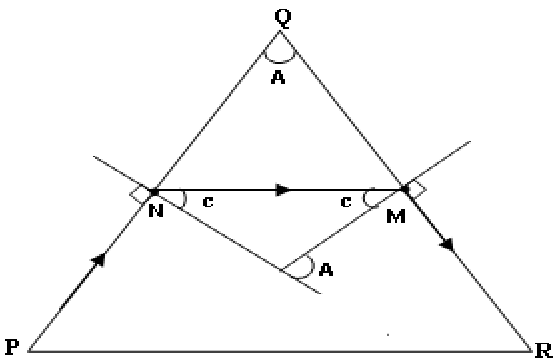
- Light suffers maximum deviation, $\boldsymbol{d_{max}} = \boldsymbol{i} + 90 - \boldsymbol{A}$
- $\cap = \sqrt{1 + \left(\frac{\sin i + \cos A}{\sin A}\right)^2}$

Conditions for maximum deviation:
✓ Either, the angle of incidence is $90^0$.

- Or, the angle of incidence results into grazing emergence. i.e, when the angle of emergence is $90^0$.

**Grazing Incidence - grazing emergence**

With grazing incidence – grazing emergency, the angle of incidence is equal to the angle of emergency which is $90^0$.




From the diagram;

$$\boldsymbol{A} = \boldsymbol{c} + \boldsymbol{c} \Leftrightarrow \boldsymbol{A} = 2\boldsymbol{c}.$$

Thus in this case;

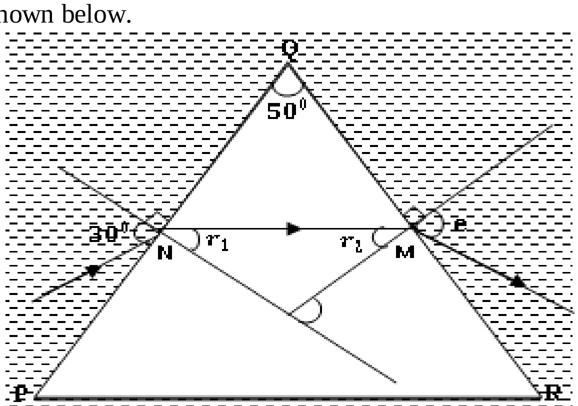
- The angle of the prism is twice the critical angle, and it is called the limiting angle of the prism.

**Note: Limiting angle of a prism:** Is the maximum angle of the prism for which the ray is emergent.

OR It is the largest angle of the prism for which both grazing incidence and grazing emergence occurs.

**Example:1**

A ray of light propagating in a liquid is incident on a prism of refractive angle $50^0$ and refractive index 1.6, at an angle of $30^0$ as shown below.




If light passes through the prism symmetrically, calculate the;

(i). Refractive index of the liquid.

(ii). Angle of deviation.

**Solution.**

**(i)**

Applying Snell's law at N:

$\cap_L \sin i = \cap \sin r_1$

$\cap_L \sin 30 = 1.6 \sin r_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots\ldots (i)$

Applying Snell's law at M:

$\cap \sin r_2 = \cap_L \sin e$

$1.6 \sin r_2 = \cap_L \sin e \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(ii)$

But, also;

$r_1 + r_2 = A$

$r_1 + r_2 = 50 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots.(iii)$

But since light passes through the prism symmetrically, then;

❖ $r_1 = r_2 = r$; Thus $2r = 50 \Leftrightarrow r = 25^0 \Leftrightarrow r_1 = r_2 = 25^0$

❖ $e = i \Leftrightarrow e = 30^0$

Thus from equation (i);

$\cap_L \sin 30 = 1.6 \sin r_1$

$\cap_L \sin 30 = 1.6 \sin 25^0$

$$\cap_L = \frac{1.6 \sin 25^0}{\sin 30} = 1.35$$

**(ii).**

$d = (i_1 + i_2) - A$

$d = (i + e) - 50$

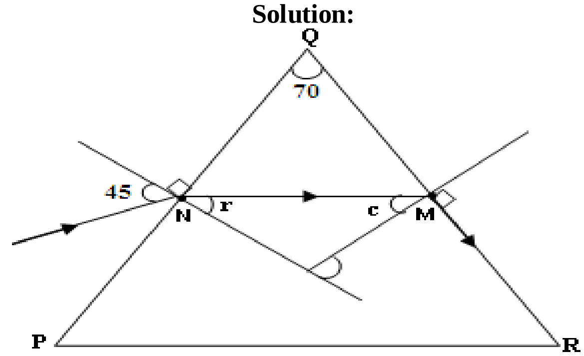
$d = (30 + 30) - 50$

$d = 10^0$

**Example:2**

Monochromatic light is incident at an angle of $45^0$ on a glass prism of refracting $70^0$ in air. The emergent light grazes the other surface of the prism. Find the refractive index of the glass prism.

**Solution:**




Applying Snell's law at N:

$\cap_a \sin i = \cap \sin r$

$\sin 45 = \cap \sin r \ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots\ldots (i)$

Applying Snell's law at M:

$\cap \sin c = \cap_a \sin e$

$\cap \sin c = \sin 90$

$\cap \sin c = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots.... (ii)$

But, also;

$r_1 + r_2 = A$

$r = 70 - c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..\ldots.(iii)$

Substituting for, r in equation (i) gives;

$\sin 45 = \cap \sin(70 - c)$
$\sin 45 = \cap (\sin 70 \cos c - \sin c \cos 70)$
$\sin 45 = \cap \sin 70 \cos c - \cap \sin c \cos 70$

But from equation (ii)

$$\cap \sin c = 1 \Leftrightarrow \cos c = \sqrt{1 - \left(\frac{1}{\cap}\right)^2}$$

$\sin 45 = \cap \sin 70 \cos c - \cos 70$

$$\frac{\sin 45 + \cos 70}{\sin 70} = \cap \sqrt{1 - \left(\frac{1}{\cap}\right)^2}$$

$$1.1165 = \cap \sqrt{1 - \left(\frac{1}{\cap}\right)^2}$$

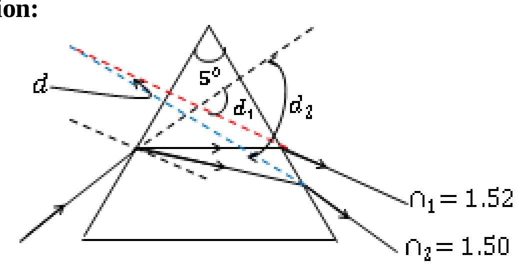
Squaring both sides gives;

$$(1.1165)^2 = \cap^2 \left[1 - \left(\frac{1}{\cap}\right)^2\right]$$

$1.2466 = \cap^2 - 1$
$\cap^2 = 2.2466$
$\cap = 1.4989$

**Example:3**
Light of two wave length is incident at a small angle on a thin prism of refracting angle $5^0$ and refractive indices 1.52 and 1.50 for the two wave lengths. Find the angular separation of the two wave length after refraction by the prism.

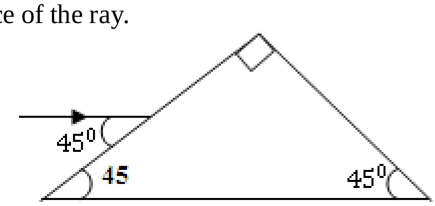
**Solution:**



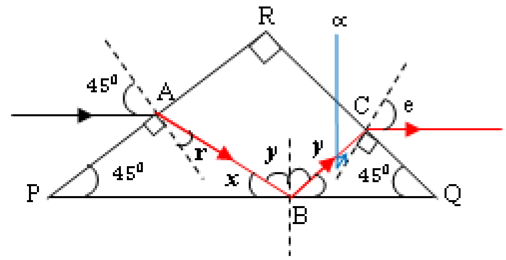

For a small angled prism,
$d = A(\cap - 1)$
$\Leftrightarrow d_1 = A(\cap_1 - 1) \Leftrightarrow d_1 = 5^0(1.52 - 1) = 2.6^0$
$\Leftrightarrow d_2 = A(\cap_2 - 1) \Leftrightarrow d_2 = 5^0(1.50 - 1) = 2.5^0$

Angular seperation, $d = d_1 - d_2$
$d = 2.6^0 - 2.5^0$
$d = 0.1^0$

**Example:4**
The figure below shows a ray of light incident on a glass prism of refractive index 1.5. Calculate the angle of emergence of the ray.




**Solution:**




At point A;
Applying Snell's law at A:
$\cap_a \sin i = \cap \sin r$
$\sin 45 = 1.5 \sin r$
$r = \sin^{-1}\left(\frac{\sin 45}{1.5}\right) = 28.1^0$

From triangle PAB:
$45 + (90 + r) + x = 180$
$45 + (90 + 28.1) + x = 180$
$45 + (118.1) + x = 180$
$163.1 + x = 180$
$x = 16.9^0$

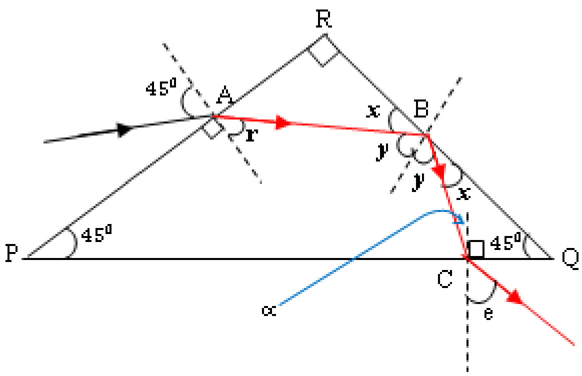
But also;
$x + y = 90$
$16.9^0 + y = 90$
$y = 73.1^0$

At point B;
Check the critical angle of the prism to see whether refraction will occur or otherwise, total internal reflection.
$\cap_g \sin y = \cap_a \sin r$
$1.5 \sin c = 1.\sin 90$
$1.5 \sin c = 1$
$c = \sin^{-1}\left(\frac{1}{1.5}\right) = 41.8^0$
Since the angle of incidence, $y = 73.1^0$ is greater than the critical angle, $c = 41.8^0$, then total internal reflection will occur at B.

From triangle BQC:
$45 + (90 + \propto) + x = 180$
$45 + (90 + \propto) + 16.9^0 = 180$
$151.9 + \propto = 180$
$\propto = 28.1^0$

At point C;
Applying Snell's law at C:
$\cap_g \sin \propto = \cap_a \sin e$
$1.5 \sin 28.1^0 = \sin e$
$e = \sin^{-1}(1.5 \sin 28.1^0) = 44.95^0$
$e = 45^0$

**Alternatively:**
Suppose, light after refraction at A strikes surface RQ.




At point A;
Applying Snell's law at A:

$\cap_a \sin i = \cap \sin r$

$\sin 45 = 1.5 \sin r$

$r = \sin^{-1}\left(\frac{\sin 45}{1.5}\right) = 28.1^0$

From triangle ARB:

$90 + (90 - r) + x = 180$

$90 + (90 - 28.1) + x = 180$

$151.9 + x = 180$

$x = 28.1^0$

But also;

$x + y = 90$

$28.1^0 + y = 90$

$y = 61.9^0$

At point B;
Check the critical angle of the prism to see whether refraction will occur or otherwise, total internal reflection.

$\cap_g \sin y = \cap_a \quad \sin r$

$1.5 \sin c = 1. \sin 90$

$1.5 \sin c = 1$

$c = \sin^{-1}\left(\frac{1}{1.5}\right) = 41.8^0$

Since the angle of incidence, $y = 61.9^0$ is greater than the critical angle, $c = 41.8^0$, then total internal reflection will occur at B.

From triangle BQC:

$45 + (90 + \propto) + x = 180$

$45 + (90 + \propto) + 28.1^0 180$

$163.1 + \propto = 180$

$\propto = 16.9^0$

At point C;
Applying Snell's law at C:

$\cap_g \sin \propto = \cap_a \sin e$

$1.5 \sin 16.9^0 = \sin e$

$e = \sin^{-1}(1.5 \sin 16.9^0) = 25.85^0$

$e = 25.9^0$

**Example:5**
A ray of light is incident on one face of an equilateral prism of refractive index1.58. Calculate the:

(i). Angle of minimum deviation.
(ii). Angle of incidence for which it occurs.

**Solution:**
$Given: A = 60^0, \quad \cap_g = 1.58, \quad D_{min} = ?$

(i)
At minimum deviation:

$$\cap = \frac{\sin\left(\frac{D_{min}+A}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

$$1.58 = \frac{\sin\left(\frac{D_{min}+60^0}{2}\right)}{\sin\left(\frac{60^0}{2}\right)} \Leftrightarrow 1.58 = \frac{\sin\left(\frac{D_{min}+60^0}{2}\right)}{\sin(30)}$$

$$1.58 \sin(30) = \sin\left(\frac{D_{min} + 60^0}{2}\right)$$

$$\frac{D_{min} + 60^0}{2} = \sin^{-1}[1.58 \sin(30)]$$

$D_{min} = 44.4^0$

(ii)

$D_{min} = 2i - A$

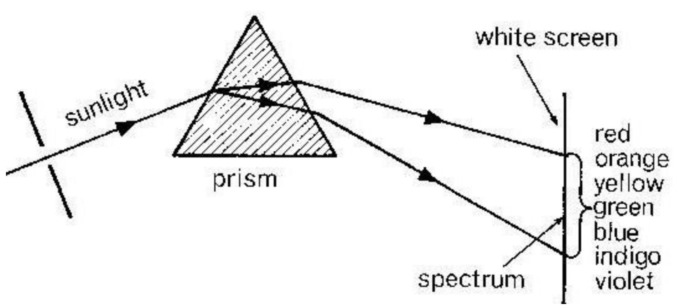
$44.4^0 = 2i - 60^0$

$2i = 15.6^0$

$i = 7.8^0$

## DISPERSION

Dispersion is the splitting or separation of white light into its component colours as it moves from one medium to another of different optical densities (refractive indices).
The colours are red, orange, yellow, green, blue, indigo, and violet. The bundle of colour formed is called a spectrum. Visible light spectrum can be made by passing a beam of white light through a glass prism.




Dispersion occurs because each colour is refracted in glass by different amount i.e. each colour has different refractive index (or different speed or different wave length) for the same glass material. So red is refracted least and violet is refracted most.

<u>Deviation of light</u>
If the prism is a small angled prism, the angle of incidence is small thus the deviation produced is given by;

$\boldsymbol{d = A(\cap - 1)}$

But red and blue are the two visible extreme colours in the spectrum of white light.
The deviation for Red is: $d_{red} = A(\cap_{red} - 1)$
The deviation for Blue is: $d_{blue} = A(\cap_{blue} - 1)$

Mean deviation of light
Mean deviation, $d_{mean} = A(\cap_{mean} - 1)$

Where; $\cap_{mean} = \dfrac{\cap_{blue} + \cap_{red}}{2}$

Mean deviation of white light:
This is equal to the deviation of yellow light.
Mean deviation, $d_{mean} = d_{yellow} = A(\cap_{yellow} - 1)$

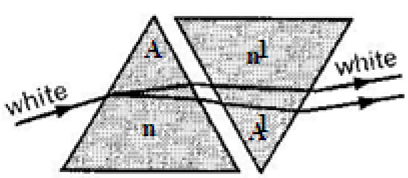
Angular dispersion of white light produced by the prism:
Angular dispersion, $\boldsymbol{d = d_b - d_r}$
$\boldsymbol{d = A(\cap_{blue} - \cap_{red})}$

Dispersive power of a material of the prism, $\omega$:

$$\text{Dispersive power} = \frac{\text{Angular dispersion}}{\text{Mean deviation}}$$

$$\boldsymbol{\omega} = \frac{\mathrm{d}}{\mathbf{d_{mean}}} = \frac{\mathbf{d_b - d_r}}{\mathbf{d_{yellow}}} = \frac{A(\cap_{blue} - \cap_{red})}{A(\cap_{yellow} - 1)}$$
$$= \frac{(\boldsymbol{\cap_{blue}} - \boldsymbol{\cap_{red}})}{(\boldsymbol{\cap_{yellow}} - 1)}$$

**Recombination of the spectrum:**
The colours of the spectrum can be recombined by;
Arranging a second prism of a different material so that the light is deviated in the opposite direction.




The first prism may be made of crown glass and it disperses the white light. The second prism may be made of flint glass and of suitable refracting angle, $A^1$. It is inverted and causes the spectrum incident on it to emerge in parallel direction if not together. Hence eliminating dispersion.

In this case, light is deviated without dispersing it. Such a pair of prisms which eliminate dispersion between two colours is called an achromatic prism for those colours.

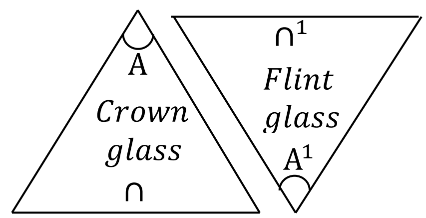
Angular dispersion produced by the first prism:
$d = d_b - d_r = A(\cap_{blue} - \cap_{red})$

Angular dispersion produced by the second prism:
$\mathrm{d'} = \mathrm{d'_b} - \mathrm{d'_r} = \mathrm{A'}(\cap'_{\mathrm{blue}} - \cap'_{\mathrm{red}})$

For no net dispersion,
$\mathrm{d} + \mathrm{d'} = 0$
$\mathrm{A}(\cap_{\mathrm{blue}} - \cap_{\mathrm{red}}) + \mathrm{A'}(\cap'_{\mathrm{blue}} - \cap'_{\mathrm{red}}) = 0$
$\mathbf{A}(\cap_{\mathbf{blue}} - \cap_{\mathbf{red}}) = -\mathbf{A'}(\cap'_{\mathbf{blue}} - \cap'_{\mathbf{red}})$

**Example:**
The refractive indices of crown glass for red and blue light respectively are 1.5146 and 1.5233 while the corresponding values for the flint glass are 1.6224 and 1.6385. Calculate the angle of flint glass prism which when combined with the crown glass prism of refracting angle $15^0$ produces deviation without dispersion for blue and red light. What deviation will the compound prism produce?

**Solution:**
**(i)**




$d_{crown} = d_b - d_r = A(\cap_{blue} - \cap_{red})$
$d_{crown} = 15^0(1.533 - 1.5146)$
$d_{crown} = 0.1305^0$

$d_{Flint} = d'_b - d'_r = A'(\cap'_{blue} - \cap'_{red})$
$d_{Flint} = A'(1.6385 - 1.6224)$
$d_{Flint} = 0.0161A'$

For no net dispersion, Angular dispersion is zero.
$d_{crown} + d_{Flint} = 0$
$0.1305^0 + 0.0161A' = 0$
$A' = -8.1^0$
**(ii)**

Deviation produced by the compound prism,
$d = d_{mean} + d'_{mean}$
$d = A(\cap_{mean} - 1) + A'(\cap'_{mean} - 1)$

$$\cap_{mean} = \frac{\cap_{blue} + \cap_{red}}{2} = \frac{1.5233 + 1.5146}{2}$$
$$= 1.51895$$
$$\cap'_{mean} = \frac{\cap'_{blue} + \cap'_{red}}{2} = \frac{1.6224 + 1.6385}{2} = 1.63045$$

$d = A(\cap_{mean} - 1) + A'(\cap'_{mean} - 1)$
$d = 15^0(1.51895 - 1) + (-8.1)(1.63045 - 1)$
$d = 2.68^0$
**Exercise:**
1. A ray of light is incident in air at an angle of $40^0$ to the normal at one face of an equilateral glass prism of refractive index 1.5. Calculate the angle through which

the ray has been deviated by the time it emerges from the prism.[$r_1 = 25.4^0$, $r_2 = 34.6^0$, $i_2 = 58.4^0$, $d = 38.4^0$]

2. A ray of light is incident on a prism of refractive index 1.3 and refracting angle $72^0$. The ray emerges from the prism at an angle of $43^0$. Find the:
   (i). Angle of incidence.[$i = 57.4^0$]
   (ii). Deviation of the ray.[$d = 28.4^0$]

3. A glass prism of refractive index 1.5 and refracting angle $60^0$ is completely immersed in a liquid of refractive index 1.3. If a ray of light passes symmetrically through the prism, calculate the;
   (i). Angle of incidence. [$i = 35.2^0$]
   (ii). Angle of deviation. [$d = 10.4^0$]

4. A ray of monochromatic light enters one face of an equilateral glass prism and is totally internally reflected at the other face.
   (i). Draw a ray diagram to show the path of light through the prism.
   (ii). Calculate the angle of incidence at the first face, if the refractive index of the glass prism is 1.53 and the angle of incidence at the second face is $42^0$.[$i = 28.2^0$]

5. A ray of light is incident on one face of an equilateral glass prism of refractive index 1.58. Calculate the;
   (i). Angle of minimum deviation.[$D_{min} = 44.4^0$]
   (ii). Angle of incidence if the ray undergoes minimum deviation.[$i = 52.2^0$]

6. Monochromatic light is incident on one face of a prism of refracting angle $60^0$, made of glass of refractive index 1.50. Calculate the least angle of incidence for the ray to emerge through the second refracting face.

[Hint; Grazing Emergence: $i_{min} = 27.9^0$]

7. A glass prism of refracting angle $60^0$ and refractive index 1.5 is immersed in a liquid of refractive index 1.4. Light is incident on it at an angle of, $i^0$. The angle I is varied until the angle, d through which the light is deviated by the prism is minimum. Find the;
   (i). Minimum value of d..[$d_{min} = 4.8^0$]
   (ii). Value of , ***i*** when d is minimum.[$i = 32.4^0$]

8. A beam of monochromatic light is incident normally on a refracting surface of a $60^0$ glass prism of refractive index 1.624. Calculate the deviation caused by the prism.
   [$d = 60^0$]

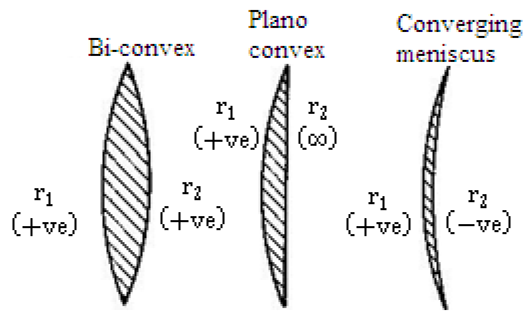
9. UNEB 2015. (c).

## LENSES

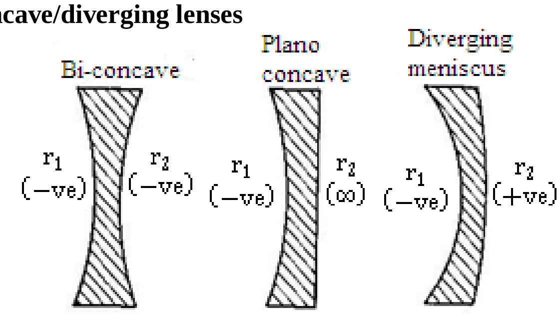
A lens is a piece of transparent material (glass or plastic) with one or more curved surfaces.

These are two types:
(i) Convex/converging lenses
(ii) Concave/diverging lenses

**Convex/converging lenses**

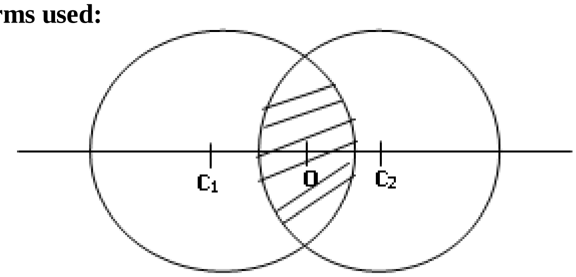



**Concave/diverging lenses**




Convex lenses are thicker in the middle than at edges while concave lenses are thinker in the edges than at the middle.

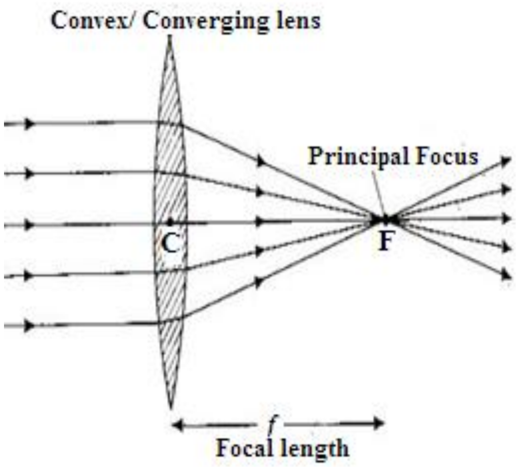
**Terms used:**



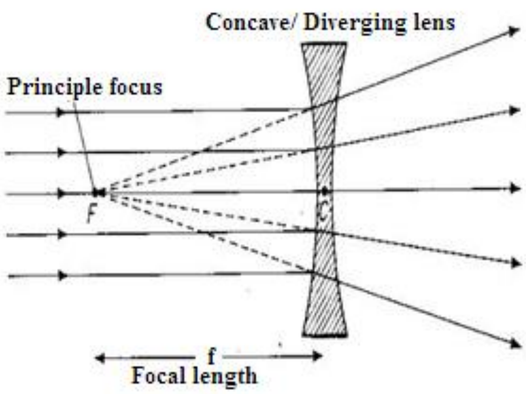

1. <u>Principal axis</u> is an imaginary line joining the principal focus (or the centres of curvature) and the optical Centre.

.

2. (a) <u>Principal focus of a convex lens</u> is a point on the principal axis to which all rays originally parallel and close to the principal axis converge after refraction by the lens.

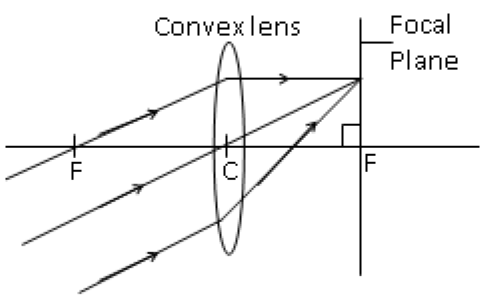



(b) Principal focus of a concave lens: This is a point on the principal axis to which all rays originally parallel and close to the principal axis appear to diverge after refraction.




**Note:** A lens has two principal foci one on each side and these are equidistant from the optical centre.
A convex lens has **real foci** while a concave lens has **virtual foci**

3. Focal Plane: Is a plane passing through the principal focus and it is perpendicular to the principal axis.




4. Focal length, *f:* This is the distance between the principal focus and the optical centre.

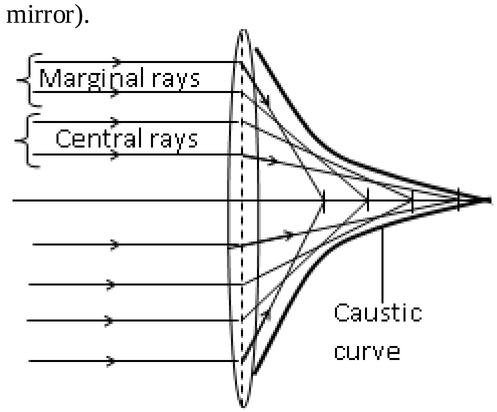
5. Optical centre, C: This is the centre of the lens at which rays pass un deviated. Or: It is point on the centre of the lens and at the principle axis through which rays incident on the lens pass un deviated.

6. Aperture: Is the geometrical width of the lens.

7. Centres of curvature, 2F: Is the centre of the sphere of which the lens surfaces form part. OR It is a point on the principle axis where any ray through it hits the lens at right angles.

8. Radius of curvature: Is the radius of the sphere of which the lens forms part. OR It is the distance between the optical centre and the centre curvature of the lens.
*Centres of the sphere of which the spherical surfaces of the lens are part of principal axis.*

9. Caustic Curve or caustic surface. Is a thick surface formed in front of a lens (of mirror), as a result of all the rays of light incident on the lens (of mirror) failing to converge at the same principle focus after refraction through the lens (or mirror).




**10. Secondary axes:**
These are rays of light that are parallel to the principle axis. They are grouped into two:
(i). Marginal rays: These are rays which are parallel and far away from the principle axis.

(ii). Central rays: These are rays which are parallel and close to the principle axis.

**Paraxial rays:** Rays which are close to principal axis and make small angles with it are called paraxial rays.

**Power of the lens**
The power, P, of the lens is the reciprocal of the focal length of the lens measured in metres.

$$\text{Power} = \frac{1}{\text{Focal length in metres}}$$

$$\mathbf{P} = \frac{1}{\mathbf{f}(\mathbf{m})}$$

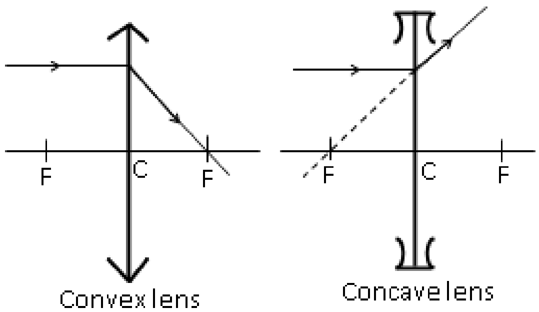
Units of power of a lens are **Dioptres (D).**

The power of a converging lens is positive while that of a diverging lens is negative.

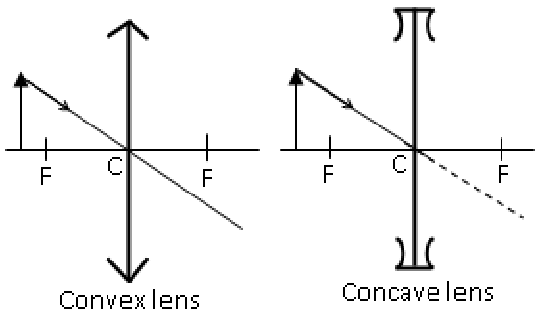
**CONSTRUCTION OF RAY DIAGRAM**

In constructing ray diagram, 2 of the 3 principal rules are used.

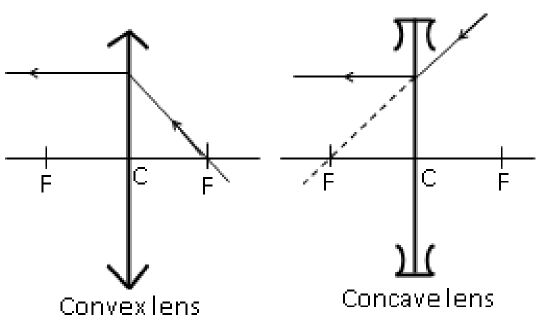
1. A ray parallel to the principal axis is refracted through the focal point.




2. A ray through the optical centre passes un deviated i.e. is not refracted.



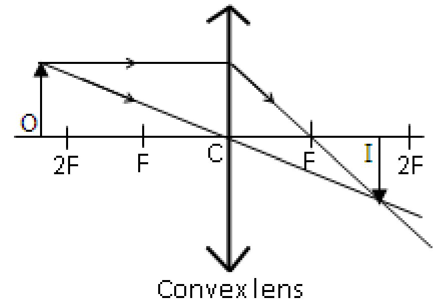

3. A ray through the principal focus emerge parallel to the principal axis after refraction.




**Images formed by convex lenses**:

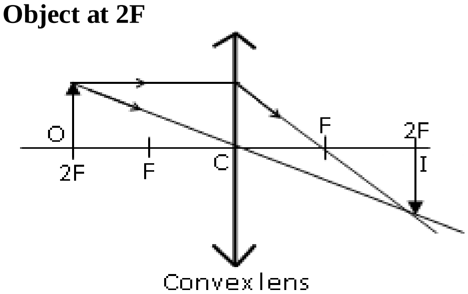
The nature of the image formed in a convex lens depends on the position of the object from the lens.

(a) **Object beyond 2F**



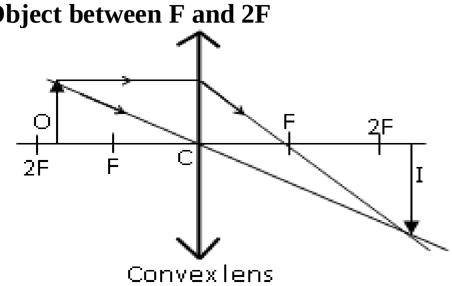

Characteristics of the image:
- Nature: Real and Inverted.
- Position: Between F and 2F.
- Magnification: Diminished

(b) **Object at 2F**



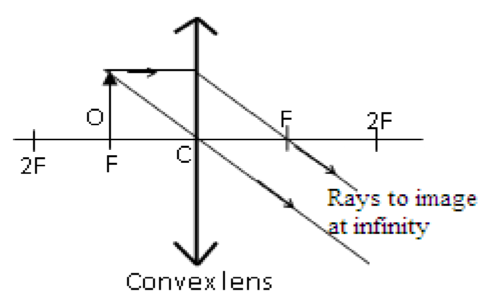

Characteristics of the image:
- Nature: Real and Inverted.
- Position: At 2F.
- Magnification: Same size as object.

(c) **Object between F and 2F**




Characteristics of the image:
- Nature: Real and Inverted.
- Position: Beyond 2F.
- Magnification: magnified.

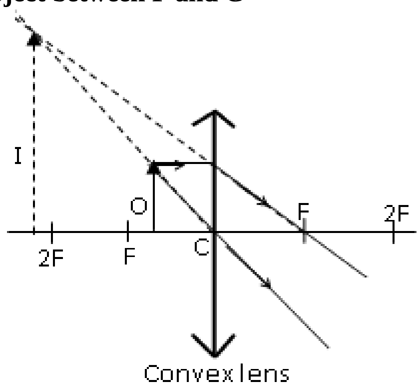
(d) **Object at F**



Characteristics of the image:

- Nature: Real and Inverted.
- Position: At infinity.
- Magnification: magnified.

**(e) Object between F and C**




Characteristics of the image:

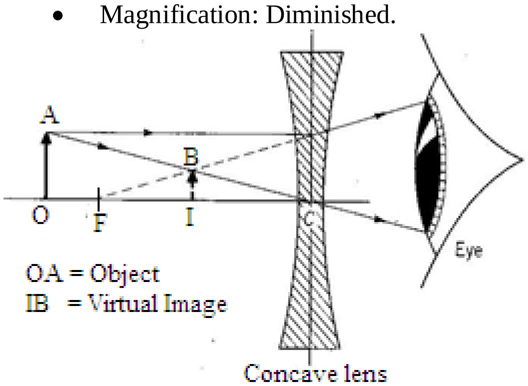
- Nature: Virtual and Upright or erect.
- Position: On the same side as the object.
- Magnification: magnified.

When the object is placed between F and C, the image is magnified and this is why the convex lens is known as a magnifying glass.

**Image Formation in a Concave Lens**

Irrespective of the position of the object, a concave lens forms an image with the following characteristics:

- Nature: Virtual and Upright or erect.
- Position: Between F and C.
- Magnification: Diminished.



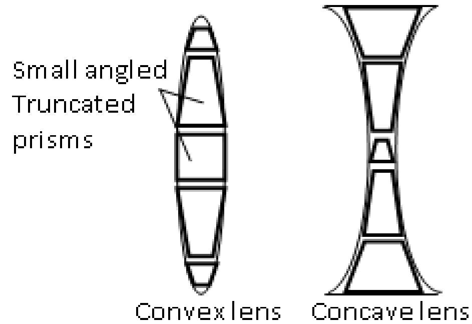

**Explanation of Action of the lens**

A thin lens is regarded as made up of a large number of small angle prisms whose angles increase from zero at the middle of the lens to a small value at the edge.

The deviation by a small angle prism is given by;

$\mathbf{d} = \mathbf{A}(\cap - 1)$. Where n is the refractive index.

Therefore, for light incident on any part of convex lens near the apex is deviated more than light incident near the middle part of the lens.




For a concave lens the transparent prism point towards the centre of the lens

The one which is far away has a longer angle of deviation

As you move to the centre, the angle, ***d*** decreases.

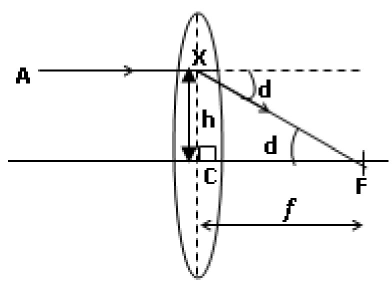
Light incident on the point of lens near the middle of the lens is deviated less than light incident on the point of lens further away from the middle of the concave lens.

## THIN LENS FORMULA

**(a) Convex lens**

**(i) Using a point Object.**

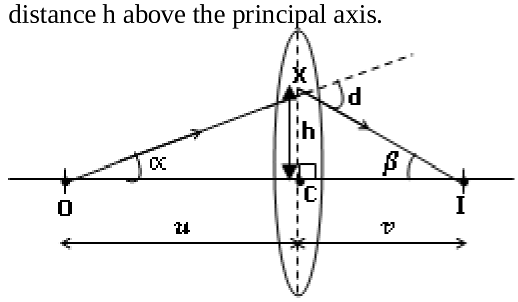
Consider array of light incident on a lens close to its principal axis and parallel to it.




For paraxial rays and if d is a small angle in radians, then;

$$d \approx \tan d = \frac{\mathrm{h}}{\mathrm{f}} \Leftrightarrow d = \frac{\mathrm{h}}{\mathrm{f}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (i)$$

Consider the formation of a real image by a real object. Also consider a ray from the object incident on the lens at a point distance h above the principal axis.




Since the lens is assumed to be composed of small angled prisms, deviation of the incident light is independent of the angle of incidence. Light is deviated through the same angle *d*, (They give a constant deviation, d).

Where, $d = A(\cap - 1)$.This follows because light is incident on the same point of the lens in both cases:

From the geometry of the diagram,
$d = \propto + \beta$……………………………..…..….. (ii)
(Exterior angle properties of a triangle).

But for paraxial rays and small angles of $\propto$ and $\beta$ in radians,

$$\left.\begin{aligned} \propto &\approx \tan\propto = \frac{h}{u} \\ \beta &\approx \tan\beta = \frac{h}{v} \end{aligned}\right\} \text{……………………………..(iii)}$$

Putting equation (iii) into equation (ii) gives;

$d = \frac{h}{u} + \frac{h}{v}$ …………………………………..(iv)

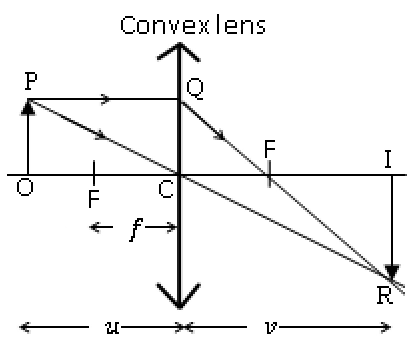
Putting equation (iv) into equation (i) gives;

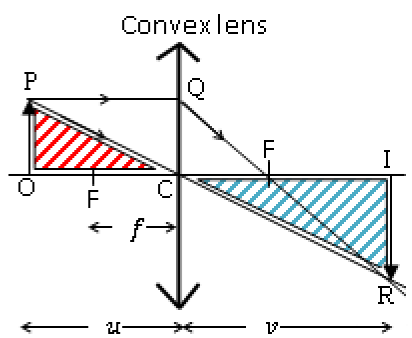
$$\frac{h}{f} = \frac{h}{u} + \frac{h}{v} \quad \Leftrightarrow h\left(\frac{1}{f}\right) = h\left(\frac{1}{u} + \frac{1}{v}\right)$$

$$\frac{1}{\mathbf{f}} = \frac{1}{\boldsymbol{u}} + \frac{1}{\boldsymbol{v}}$$

This is the lens formula.

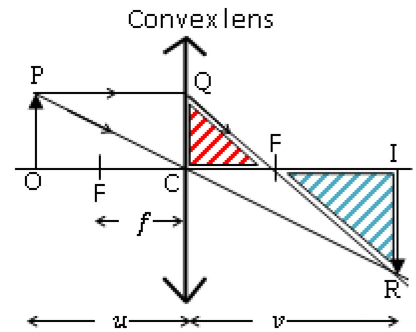
**(ii) Using a finite Object.**




Triangles $\triangle$ OPC and $\triangle$ IRC are similar.

$$\frac{IR}{OP} = \frac{IC}{OC} = \frac{v}{u} \text{………………………….(i)}$$




Triangles $\triangle$ CQF and $\triangle$ IRF are similar.

$$\frac{IR}{CQ} = \frac{IF}{CF} = \frac{v - f}{f}$$

$$IF = (IC - FC) = v - f$$

$$\frac{IR}{CQ} = \frac{v - f}{f}$$

But CQ = OP

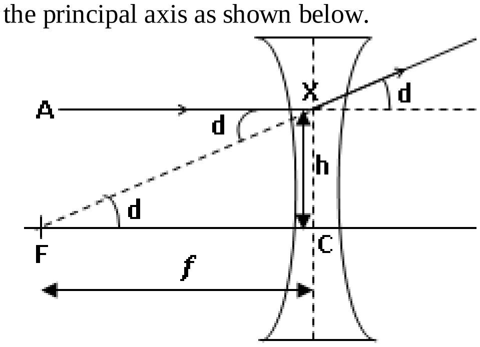
$$\Leftrightarrow \frac{IR}{OP} = \frac{v - f}{f} \text{………………………….(ii)}$$

From equation (i) and (ii)

| | |
|---|---|
| $\frac{IR}{OP} = \frac{v}{u} = \frac{v-f}{f}$<br>$\frac{v}{u} = \frac{v-f}{f}$<br>$vf = u(v - f)$<br>$vf = uv - uf$<br>$uv = vf + uf$ | Divide all through by uvf<br>$\frac{uf}{uvf} + \frac{vf}{uvf} = \frac{uv}{uvf}$<br>$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$<br>$\frac{1}{\mathbf{f}} = \frac{1}{\mathbf{u}} + \frac{1}{\mathbf{v}}$ |

**(b) Concave lens**

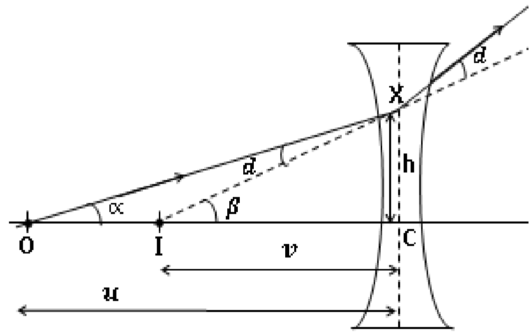
Consider a ray AX incident on a concave lens parallel and close to the principal axis as shown below.




For paraxial rays and if d is a small angle in radians, then;

$$d \approx \tan d = \frac{h}{f} \Leftrightarrow d = \frac{h}{f} \text{………………………….}(i)$$

Consider a point object; O forming a virtual image, I. let the ray strike the lens at a point X, a distance h above the axis of the lens.




From the geometry of the diagram,

$\beta = \propto + d$……………………………..…..….. (ii)

(Exterior angle properties of a triangle).

But for paraxial rays and small angles of $\propto$ and $\beta$ in radians,

$$\left.\begin{aligned} \propto &\approx \tan\propto = \frac{h}{u} \\ \beta &\approx \tan\beta = \frac{h}{v} \end{aligned}\right\} \quad \text{……………………………..(iii)}$$

Putting equation (iii) into equation (ii) gives;

$\frac{h}{v} = \frac{h}{u} + d$ …………………………………..(iv)

Putting equation (iv) into equation (i) gives;

$$\frac{\mathrm{h}}{\mathrm{v}} = \frac{h}{u} + \frac{h}{f}$$

Considering real is positive, virtual is negative sign convention, the principal focus and image of a concave lens are virtual hence f is negative and v is negative.

$$\Leftrightarrow \mathrm{h}\left(\frac{1}{-\mathrm{f}}\right) = \mathrm{h}\left(\frac{1}{-v} - \frac{1}{u}\right)$$

$$\Leftrightarrow -\mathrm{h}\left(\frac{1}{\mathrm{f}}\right) = -\mathrm{h}\left(\frac{1}{v} + \frac{1}{u}\right)$$

$$\frac{1}{\mathbf{f}} = \frac{1}{\boldsymbol{u}} + \frac{1}{\boldsymbol{v}}$$

This is the lens formula.

**Exercise:**

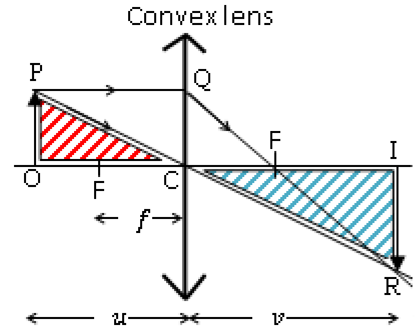
*Following the example for mirrors, derive the lens formula using a finite object and a concave lens.*

**Linear and lateral or transverse magnification**

It is the ratio of the height of the image to the height of the object.

$$\textbf{Magnification; m} = \frac{\textbf{Image Height}}{\textbf{Object Height}}$$

Consider the formation of an image of finite object by a convex lens.




Triangles $\triangle$ OPC and $\triangle$ IJP are similar.

$$\frac{\mathrm{IR}}{\mathrm{OP}} = \frac{\mathrm{IC}}{\mathrm{OC}} = \frac{\mathrm{v}}{\mathrm{u}}$$

$$\mathrm{m} = \frac{\mathrm{IR}}{\mathrm{OP}} = \frac{\text{Image Height}}{\text{Object Height}}$$

$$\mathrm{m} = \frac{\mathrm{IC}}{\mathrm{OC}} = \frac{\text{Image distance from lens}}{\text{Object distance from lens}}$$

**Alternatively, you can use:**

$\text{Angle(PCO)} = \text{Angle(ICR)} \rightarrow \text{Vert. Opp. Angles}$

$$\tan(\mathrm{PCO}) = \tan(\mathrm{ICR})$$

$$\frac{\mathrm{h_0}}{\mathrm{u}} = \frac{\mathrm{h_I}}{\mathrm{v}} \Leftrightarrow \frac{\mathbf{h_I}}{\mathbf{h_0}} = \frac{\boldsymbol{v}}{\mathbf{u}}$$

$$\frac{\mathbf{h_I}}{\mathbf{h_0}} = \frac{\boldsymbol{v}}{\mathbf{u}}$$

But, $\frac{\mathbf{h_I}}{\mathbf{h_0}}$ is linear or lateral or transverse magnification, **m**. Thus;

$$\mathbf{m} = \frac{\textbf{Image Height}}{\textbf{Object Height}} = \frac{\textbf{Image distance}}{\textbf{Object distance}}$$

From the mirror formula;

$$\frac{1}{\mathrm{f}} = \frac{1}{\mathrm{u}} + \frac{1}{\mathrm{v}}$$

Multiplying through by v gives;

$$\frac{\mathrm{v}}{\mathrm{f}} = \frac{\mathrm{v}}{\mathrm{u}} + 1. \quad \text{But } \frac{\mathrm{v}}{\mathrm{u}} = \mathrm{m}$$

$$\frac{\mathrm{v}}{\mathrm{f}} = \mathrm{m} + 1$$

$$\mathbf{m} = \frac{\mathbf{v}}{\mathbf{f}} - 1 \Leftrightarrow \mathbf{m} = \frac{\mathbf{v} - \mathbf{f}}{\mathbf{f}}$$

Multiplying through by u gives;

$$\frac{\mathrm{u}}{\mathrm{f}} = 1 + \frac{\mathrm{u}}{\mathrm{v}}. \quad \text{But } \frac{\mathrm{u}}{\mathrm{v}} = \frac{1}{\mathrm{m}}$$

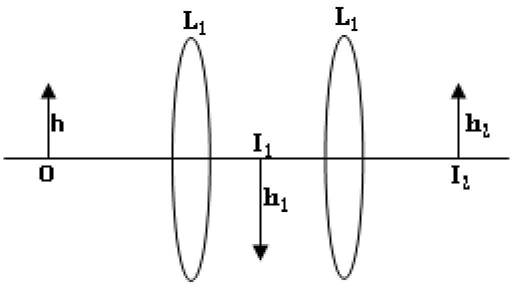
$$\frac{\mathrm{u}}{\mathrm{f}} = 1 + \frac{1}{\mathrm{m}}$$

$$\frac{1}{\mathbf{m}} = \frac{\mathbf{u}}{\mathbf{f}} - 1 \Leftrightarrow \frac{1}{\mathrm{m}} = \frac{\mathrm{u-f}}{\mathrm{f}} \Leftrightarrow \mathbf{m} = \frac{\mathbf{f}}{\mathbf{u-f}}$$

**Linear magnification of a system of lenses.**

Consider two coaxial lenses $L_1$ and $L_2$ forming images $I_1$ and $I_2$ respectively as shown below.

NOTE: Coaxially means that the lenses are on the same axis.




$$\text{Magnification; m} = \frac{\text{Final Image Height}}{\text{Object Height}}$$

$$\mathrm{m} = \frac{\mathrm{h_2}}{\mathrm{h}}$$

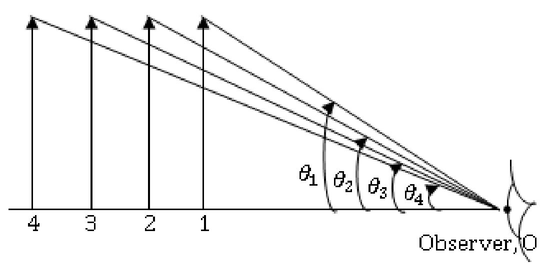
$$\mathrm{m} = \frac{\mathrm{h_1}}{\mathrm{h}} \times \frac{\mathrm{h_2}}{h_1}$$

$$\mathbf{m} = \boldsymbol{m_1 m_2}$$

In general, the linear magnification of a lens system is the product of the linear magnifications of the respective lenses in the system.

**Why an object far away from the observer appears to be smaller than when it is nearer.**

Consider four poles, of the same height placed at different positions from an observer at O.




The pole that is nearer to the observer subtends a bigger angle at the observer's eye than the furthest pole, (pole, number 4). Since the height of the object is proportional to the angle it subtends at the eye, the furthest pole appears shorter.

**Example: 1**

The magnification of a thin converging lens is m. When the lens is moved a distance, d towards the object, the magnification becomes, m'. Show that the focal length, f of the lens is given by;

$$\boldsymbol{f} = \frac{\boldsymbol{dmm'}}{\boldsymbol{m' - m}}$$

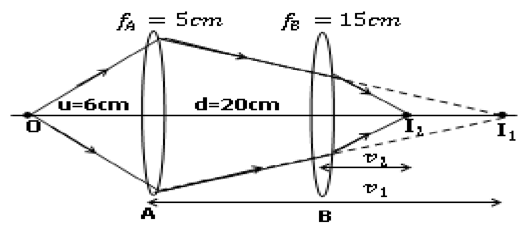
**Example: 2**

An object is placed 6cm from a thin converging lens, A of focal length 5cm. Another thin converging lens, B of focal length 15 cm is placed coaxially with A and 20cm from it on the side away from the object. Find the;

(i). Position and nature of the final image.

(ii). Magnification of the final image.

**Solutions:**




Action of lens A:

$$\frac{1}{\boldsymbol{f_A}} = \frac{1}{\boldsymbol{u}_1} + \frac{1}{\boldsymbol{v}_1}$$

$$\frac{1}{5} = \frac{1}{6} + \frac{1}{\boldsymbol{v}_1}$$

$\mathrm{v_1} = 30\mathrm{cm}$

Action of lens B:

$\mathrm{I_1}$acts as avirtual object for lens B

$\mathrm{u_2} = -(\mathrm{v_1} - \mathrm{d})$

$\mathrm{u_2} = -(30 - 20)$

$\mathrm{u_2} = -10\mathrm{cm}$

$$\frac{1}{\mathbf{f_B}} = \frac{1}{\mathbf{u}_2} + \frac{1}{\mathbf{v}_2}$$

$$\frac{1}{15} = \frac{1}{-10} + \frac{1}{\mathbf{v}_2}$$

- Thus the final image is 6cm from lens B.
- Then final image is real ( because $\mathrm{v_2}$ is positive)
- It is also magnified.

(ii)

Magnification, m:

$\mathrm{m} = \mathrm{m_A m_B}$

$$\mathrm{m} = \left(\frac{30}{6}\right)\left(\frac{6}{10}\right)$$

$\mathrm{m} = 3$

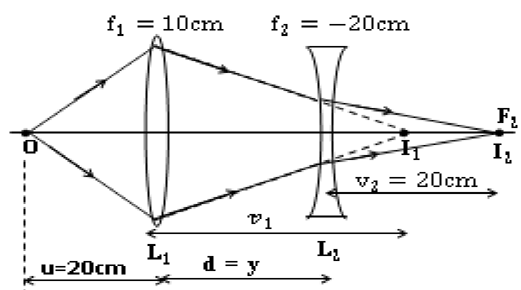
**Example: 3**

A converging lens, $L_1$ of focal length 10cm is placed at a distance, y in front of a diverging lens, $L_2$ of focal length 20 cm. An illuminated object is placed at a distance of 20 cm in front of $L_1$ and the final image by $L_2$ formed at the principal focus of $L_2$. Calculate the;

(i). Distance, y.

(ii). Final magnification.

**Solutions:**



Action of lens $\mathbf{L}_1$:

$$\frac{1}{\boldsymbol{f}_1}=\frac{1}{\boldsymbol{u}_1}+\frac{1}{\boldsymbol{v}_1}$$

$$\frac{1}{10}=\frac{1}{20}+\frac{1}{\boldsymbol{v}_1}$$

$$\mathrm{v}_1=20\mathrm{cm}$$

Action of lens $\mathbf{L}_2$:

$\mathrm{I}_1$ acts as avirtual object for lens B

$$\mathrm{u}_2=-(\mathrm{v}_1-\mathrm{d})$$
$$\mathrm{u}_2=-(20-\mathrm{y})$$
$$\mathrm{u}_2=(y-20)\mathrm{cm}$$

$$\frac{1}{\mathbf{f}_2}=\frac{1}{\mathbf{u}_2}+\frac{1}{\mathbf{v}_2}$$

$$\frac{1}{-20}=\frac{1}{\mathbf{y}-20}+\frac{1}{20}$$

$$\mathrm{y}=10\mathrm{cm}$$

(ii)

Magnification, m:

$$\mathrm{m}=\mathrm{m}_1\mathrm{m}_2$$

$$\mathrm{m}=\left(\frac{20}{20}\right)\left(\frac{20}{10}\right)$$

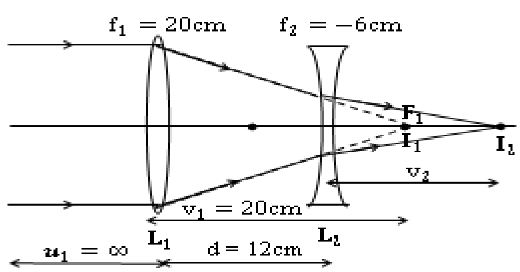
$$\mathrm{m}=2$$

**Example: 4**

A converging lens of focal length 20cm is placed coaxially with a diverging lens of focal length 6cm such that they are 12cm apart with the convex lens facing a distant object. Determine the;

(i). Position of the final image.
(ii). Nature of the final image formed.

**Solution:**




Action of lens $\mathbf{L}_1$:

$$\frac{1}{\boldsymbol{f}_1}=\frac{1}{\boldsymbol{u}_1}+\frac{1}{\boldsymbol{v}_1}$$

$$\frac{1}{20}=\frac{1}{\infty}+\frac{1}{\boldsymbol{v}_1}$$

$$\mathrm{v}_1=20\mathrm{cm}$$

Action of lens $\mathbf{L}_2$:

$\mathrm{I}_1$ acts as avirtual object for lens B

$$\mathrm{u}_2=-(\mathrm{v}_1-\mathrm{d})$$
$$\mathrm{u}_2=-(20-12)$$
$$\mathrm{u}_2=-8\mathrm{cm}$$

$$\frac{1}{\mathbf{f}_2}=\frac{1}{\mathbf{u}_2}+\frac{1}{\mathbf{v}_2}$$

$$\frac{1}{-6}=\frac{1}{-8}+\frac{1}{\boldsymbol{v}_2}$$

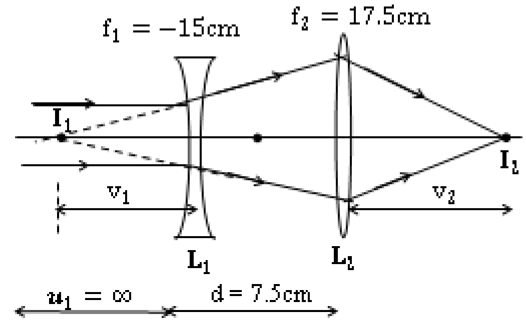
$$v_2=-24\mathrm{cm}$$

Thus the final image is virtual and 24cm behind the concave

**Example: 5**

A convex lens and a concave lens of focal length 17.5cm and 15cm respectively are mounted coaxially7.5cm apart with the concave lens facing a distant object. Find the final position of the image. [v = 78.75cm from convex lens.]

**Solution:**




Action of lens $\mathbf{L}_1$:

$$\frac{1}{\boldsymbol{f}_1}=\frac{1}{\boldsymbol{u}_1}+\frac{1}{\boldsymbol{v}_1}$$

$$\frac{1}{-15}=\frac{1}{\infty}+\frac{1}{\boldsymbol{v}_1}$$

$$\mathrm{v}_1=-15\mathrm{cm}$$

Action of lens $\mathbf{L}_2$:

$\mathrm{I}_1$ acts as avirtual object for lens B

$$\mathrm{u}_2=-(\mathrm{v}_1-\mathrm{d})$$
$$\mathrm{u}_2=-(-15-7.5)$$
$$\mathrm{u}_2=22.5\mathrm{cm}$$

$$\frac{1}{\mathbf{f}_2}=\frac{1}{\mathbf{u}_2}+\frac{1}{\mathbf{v}_2}$$

$$\frac{1}{17.5}=\frac{1}{22.5}+\frac{1}{\boldsymbol{v}_2}$$

$$v_2=78.75\mathrm{cm}$$

Thus the final image is real and 78.75cm from the convex lens.

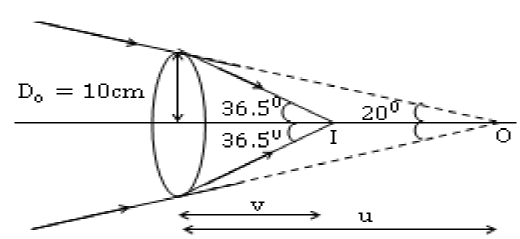
**Example: 6**

A converging beam of light in the shape of a cone, with a vertex angle of 40º falls on a circular diaphragm of diameter 10cm. When a converging lens is fixed in the diaphragm, the new vertex angle becomes 73º. Calculate the

(i) Focal length of the lens
(ii) Displacement of the vertex of the cone.

**Solution:**

**(i)**




$$\tan 20^0=\frac{10}{-u}$$

$$u=\frac{-10}{\tan 20^0}=-27.5\mathrm{cm}$$

$$\tan 36.5^0=\frac{10}{v}$$

$$v=\frac{10}{\tan 36.5^0}=13.5\mathrm{cm}$$

Then using the lens formula;

$$\frac{1}{\mathbf{f}} = \frac{1}{\boldsymbol{u}} + \frac{1}{\boldsymbol{v}}$$

$$\frac{1}{\mathbf{f}} = \frac{1}{-27.5} + \frac{1}{13.5}$$

$$\mathrm{f} = 26.5\mathrm{cm}$$

(ii) Displacement of the vertex
$d = u - v$
$d = 27.5 - 13.5$
$d = 14.0\text{cm}$

**Exercise:**

1. A converging lens forms an image of height, $\mathbf{h}_1$ on a screen, of object O of height, $\mathbf{h}$. When the lens is displaced towards the screen, an image of height $\mathbf{h}_2$is formed on the screen.
   (i). Sketch a ray diagram to show the formation of the images on the screen.
   (ii). Show that, $\mathbf{h} = \sqrt{\mathbf{h}_1\mathbf{h}_2}$.

2. A lens $L_1$ casts a real image of a distant object on a screen placed at a distance of 15cm away. When another lens $L_2$ is placed 5cm beyond lens $L_1$ the screen has to be shifted by 10cm further away to locate the real image formed. Determine the focal length and type of lens $L_2$.
   [$f_2 = -20\text{cm}$, Hecce $\mathrm{L}_2$ is concave lens.]

3. (a) Derive the relationship between the focal length f, object distance *u*, and image distance *v* for a thin lens.
   (c) A thin converging lens P, of focal length 10 cm and a thin diverging lens Q, of focal length 15cm are placed coaxially 50 cm apart. If an object O is placed 12cm from P on the side remote from Q
   (i) Find the position, nature and magnification of the final image. [$v_2 = 30\text{cm from concave lens Q}$]
   (ii) Sketch a ray diagram to show the formation of the final image.

4. Light from an object passes through a thin converging lens of focal length 15cm placed 20cm from it and then through a diverging lens of focal length 10cm placed 55cm from the converging lens. Find the;
   (i) Position of the image to the first lens. [$v_1 = 60\text{cm}$]
   (ii) Position of the image due to the lens system. [$v_2 = 10\text{cm}$]
   (iii) Linear magnification of the lens system. [M = 6]

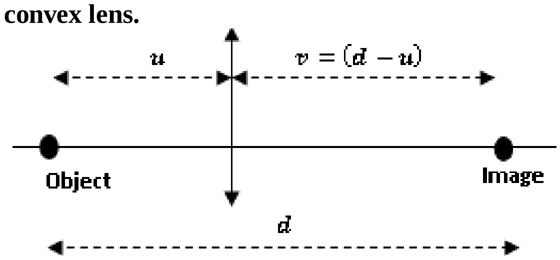
5. An object is placed at the zero centimetre mark on an optical bench and a convex lens is placed at the 50cm mark. A real image is formed at the 70cm mark. Without moving the object and the convex lens, a diverging lens is placed at the 60cm mark and a new real image is formed at the 80cm mark. Find the;
   (i) Focal length of each lens. [$f_1 = 14.3\text{cm}$, $f_2 = -20\text{cm}$ ]
   (ii) Magnification produced. [M = 0.8]

6. An insect is placed at a distance of 30cm from a converging lens of focal length 10cm. Determine the distances from the lens where a second converging lens of focal; length 40cm must be placed in order to produce an image of the same size as the object which is;
   (i) Erect. [$d = v_1 + u_2 = 75\text{cm}$ ]
   (ii) Inverted [$d = v_1 + u_2 = 35\text{cm}$ ]

7. Two bi focal lenses, each of focal length 8cm are placed 2cm apart. If the object is placed 14cm in front of one of the lenses but on the outside, find the;
   (i). Nature and position of the final image. [$v_2 = 5.41\text{cm}$ ]
   (ii). Magnification. [ M = 0.433]

8. Two lenses of focal length 4cm and 14cm respectively are placed 2cm apart. Find the image position and magnification of an object placed 6cm from the lenses, if;
   (i). The 1st lens is convex and the 2nd is concave. [$v_2 = 35\text{cm}$ ; M = 7]
   (ii). The 1st lens is concave and the 2nd is convex.
   (iii). Both lenses are convex.

**Least distance between an object and a real image formed by a convex lens.**




Suppose the object and image are a distance d apart with the image a distance x beyond the lens.

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$
$$\frac{1}{f} = \frac{1}{u} + \frac{1}{d-u}$$

$$du - u^2 = df$$
$$du - u^2 - df = 0$$
$$u^2 - du + df = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$$

This is a quadratic expression;

$$u = \frac{-(-d) \pm \sqrt{(-d)^2 - 4(1)df}}{2(1)}$$

$$u = \frac{d \pm \sqrt{d^2 - 4df}}{2}$$

For real values of the object distance, u;

$$d^2 - 4df \geq 0$$
$$d^2 \geq 4df$$
$$d \geq 4f \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (ii)$$

The least distance between an object and the real image of the object formed by the convex lens is four times its focal length, 4f.

Putting $\boldsymbol{d} = 4\boldsymbol{f}$ into equation (i) gives;

$$u = \frac{d \pm \sqrt{d^2 - 4df}}{2}$$

$$u = \frac{4f \pm \sqrt{(4f)^2 - 4(4f)f}}{2}$$

$$u = \frac{4f \pm \sqrt{16f^2 - 16f^2}}{2}$$

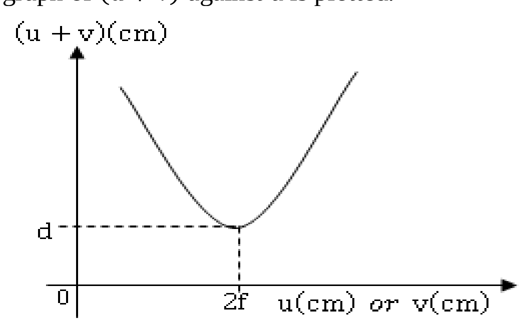
$\boldsymbol{u} = 2\boldsymbol{f}$
$\boldsymbol{v} = (d - u) = (4f - 2f) = 2\boldsymbol{f}$
Thus for an equi- convex lens, this occurs when the object is placed at the centre of curvature.

**NOTE:**
**Experimentally, we can determine the least distance between a real object and a real image as follows:**

- Given different values of the object distance, u, we determine their corresponding values of v by using, either;
  - ✓ Object pin and no parallax method, or
  - ✓ An illuminated object and screen.
- The results are tabulated including values of $(u + v)$.

| u(cm) | v(cm) | (u + v)(cm) |
|---|---|---|
| - | - | - |
| - | - | - |

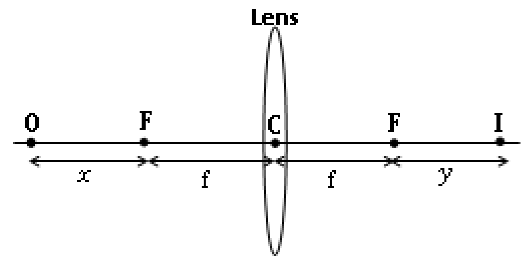
- A graph of (u + v) against u is plotted.




- The minimum value, d on the $(u + v)$-axis is read off and it is equal to least distance between a real object and a real image.
- The focal length, f of the lens used is also calculated from; $\boldsymbol{d} = 4\boldsymbol{f}$

**CONJUGATE POINTS:**
These are points on the principal axis which are equidistant and interchangeable.
Conjugate points are two points on the principal axis which are such that when an object is placed at one, an image is formed at the other.

**Newton's equation.**




Consider an object O placed at a distance, x from the principle focus of the convex lens. The lens forms a real image I, on the opposite side of the lens at a distance y from the principal focus.
When the object was placed at I, the lens would form the image of the object at O.
Then O and I are equidistant from the optical centre, and are interchangeable. These points are called conjugate points.

Object distance, $u = x + f$
Image distance, $v = y + f$

Then, from the lens formula;

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

$$\frac{1}{f} = \frac{1}{x + f} + \frac{1}{y + f}$$

$$\frac{1}{f} = \frac{2f + x + y}{(x + f)(y + f)}$$

$f^2 = xy$

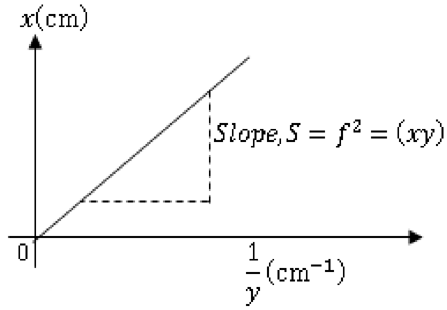
$$\boldsymbol{f} = \sqrt{\boldsymbol{xy}} = (\boldsymbol{xy})^2$$

This is Newton's equation.

**NOTE:**
The idea of conjugate points is applied in the determination of the focal length of the lens whose surfaces are not accessible. E.g a lens inside a tube.
By obtaining various distances of x and y, the results are tabulated including values of $\frac{1}{y}$.

| x(cm) | y(cm) | $\frac{1}{y}$ (cm$^{-1}$) |
|---|---|---|
| - | - | - |
| - | - | - |

A graph of x against $\frac{1}{y}$ is plotted.




It is straight line graph through the origin. Its slope is determined and the focal length of the lens used is calculated from;

$$\boldsymbol{f = \sqrt{S}}$$

**Theory:**

From: $f^2 = xy \Leftrightarrow y = \frac{f^2}{x} \Leftrightarrow y = f^2.\frac{1}{x}$

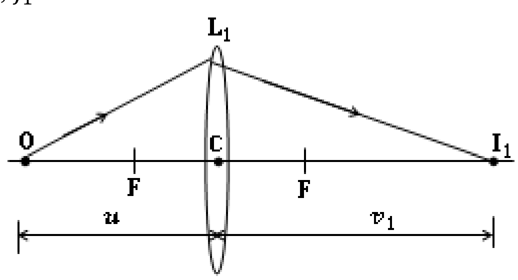
Compare with; $y = mx + c$

$\Leftrightarrow$ m = slope, $S = f^2$ and $c = 0$

**Combined focal length of two thin lenses in contact**

Action of lens $L_1$

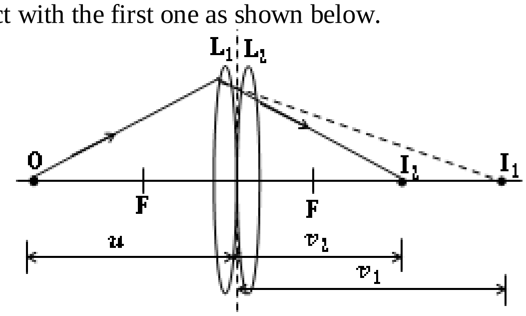
Consider formation of real image, I by a convex lens of focal length, $f_1$.




Then, from the lens formula;

$$\frac{1}{f_1} = \frac{1}{u} + \frac{1}{v_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

A second convex lens of focal length $f_2$ is now placed in contact with the first one as shown below.




Action of lens $L_2$

$\mathbf{I}_1$ acts as a virtual object for the second lens $L_2$, and so, $u_2 = -v_1$.

$$\frac{1}{f_2} = \frac{1}{-v_1} + \frac{1}{v_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Equation (i) and (ii) added together, give;

$$\frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{u} + \frac{1}{v_1} + \frac{1}{-v_1} + \frac{1}{v_2}$$

$$\frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{u} + \frac{1}{v_2}$$

Taking the combination to be equivalent to a single convex lens of focal length, F, then; $\frac{1}{\mathbf{F}} = \frac{1}{\mathbf{u}} + \frac{1}{\mathbf{v_2}}$

$$\frac{1}{\mathbf{f_1}} + \frac{1}{\mathbf{f_2}} = \frac{1}{\mathbf{F}}$$

**NOTE:**

This expression can be extended to any number, n of lenses in contact. Thus in general;

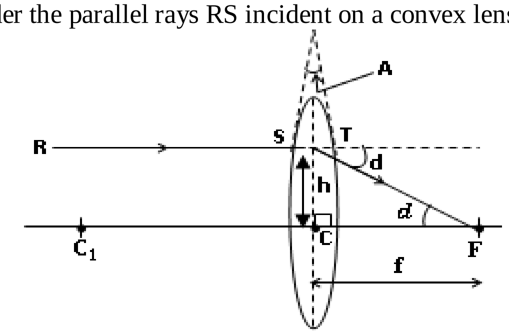
$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} + \cdots \ldots\ldots.. + \frac{1}{f_n}.$$

Where, F is the combined focal length of the lenses in contact.

**Full thin lens formula Or Lens Makers formula.**

**(**<u>Relationship between focal length, radii of curvature and refractive index of a lens).</u>

Consider the parallel rays RS incident on a convex lens.




If A is the refracting angle of the small angle prism formed by tangents at S and T, deviation. d is;

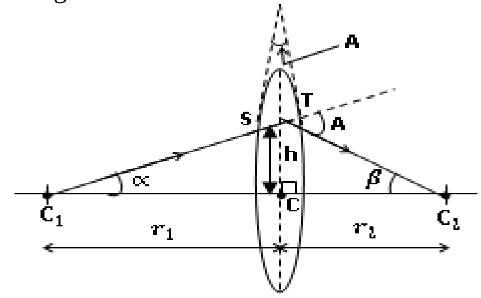
$$d \approx \tan d = \frac{h}{f} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (i)$$

For small angle prism, the deviation, d is given by;

$$d = A(\cap - 1)$$

$$\frac{h}{f(\cap - 1)} = A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (ii)$$

Consider the radii normal to the surfaces of the lens at Sand T passing through the centres of curvature $c_1$ and $c_2$



From the diagram;

$$\propto +\beta = A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$$

(Using exterior < properties)

For $\propto$ and $\beta$ small angles in radians, then;

$$\left.\begin{aligned}\propto &\approx \tan\propto = \frac{h}{r_1}\\ \beta &\approx \tan\beta = \frac{h}{r_2}\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (iv)$$

Putting equation (iv) into equation (iii);

$$\frac{h}{r_1}+\frac{h}{r_2} = A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (v)$$

Substitute equation (v) into (ii)

$$\frac{h}{r_1}+\frac{h}{r_2} = \frac{h}{f(\cap -1)}$$

$$h(\cap -1)\left(\frac{1}{r_1}+\frac{1}{r_2}\right) = h\left(\frac{1}{f}\right)$$

$$(\cap -1)\left(\frac{1}{r_1}+\frac{1}{r_2}\right) = \frac{1}{f}$$

Hence;

$$\frac{1}{\mathbf{f}} = (\cap -1)\left(\frac{1}{\mathbf{r_1}}+\frac{1}{\mathbf{r_2}}\right)$$

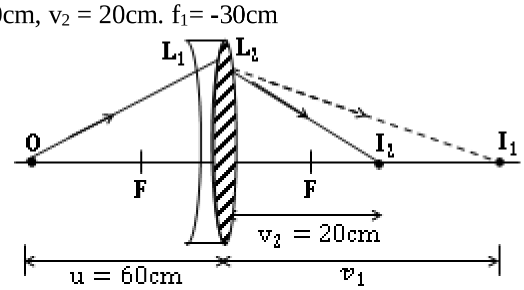
This is called the lens makers' formula. Where $\mathbf{r_1}$ and $r_2$ are the radii of curvature of the two surfaces of the lens, **f** its focal length and **n** the refractive index of the material of the lens.

**Example:1**

A small object is placed at a distance of 60cm to the left of a diverging lens of focal length 30cm. A converging lens is then placed in contact with the diverging lens. If a real image is formed at a distance of 20 cm to the right of the combined lenses; find the focal length of the converging lens.

**Solution:**

u = 60cm, $v_2$ = 20cm. $f_1$= -30cm




For the combined focal length, F:

$$\frac{1}{\mathbf{F}} = \frac{1}{\mathbf{u}}+\frac{1}{\mathbf{v_2}}$$

$$\frac{1}{F} = \frac{1}{60}+\frac{1}{20} = \frac{1}{15}$$

$$F = 15cm$$

But also, the combined focal length, F is given by:

$$\frac{1}{F} = \frac{1}{f_1}+\frac{1}{f_2}$$

$$\frac{1}{15} = \frac{1}{-30}+\frac{1}{f_2} \Leftrightarrow \frac{1}{f_2} = \frac{1}{10}$$

$$f_2 = 10cm$$

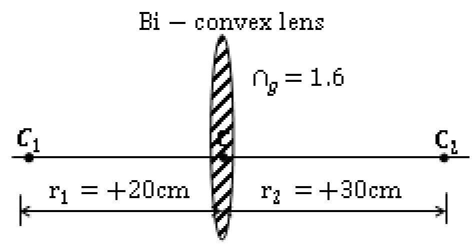
**Example: 2**

A thin lens with faces of radii of curvature 20cm and 30cm is to be made from glass with refractive index 1.6. What will be the focal length of the lens, if it is;

(i). Bi-convex
(ii). Bi-concave
(iii). Converging meniscus

**Solution:**

(i) Bi-convex;



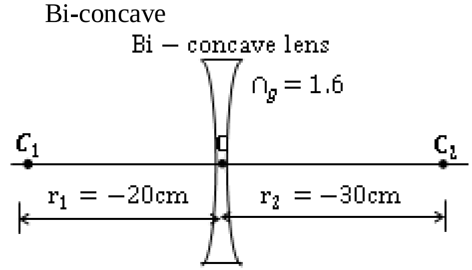

$r_1 = +20cm, r_2 = +30cm, \cap_g = 1.6$

$$\frac{1}{f_g} = (\cap_g - 1)\left(\frac{1}{r_1}+\frac{1}{r_2}\right);$$

$$\frac{1}{f_g} = (1.6-1)\left(\frac{1}{20}+\frac{1}{30}\right)$$

$$f_g = 20cm$$

(ii) Bi-concave



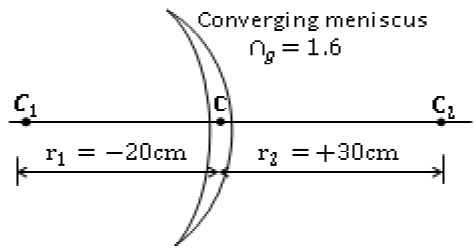

$r_1 = -20cm, r_2 = -30cm, \cap_g = 1.6$

$$\frac{1}{f_g} = (\cap_g - 1)\left(\frac{1}{r_1}+\frac{1}{r_2}\right);$$

$$\frac{1}{f_g} = (1.6-1)\left(\frac{1}{-20}+\frac{1}{-30}\right)$$

$$f_g = -20cm$$

(iii) Converging meniscus.




$r_1 = -20$cm, $r_2 = +30$cm, $\cap_g = 1.6$

$$\frac{1}{f_g} = (\cap_g - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right);$$

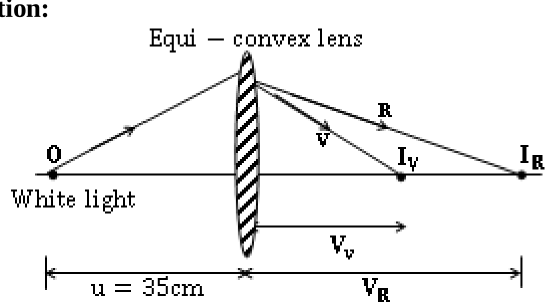
$$\frac{1}{f_g} = (1.6 - 1)\left(\frac{1}{-20} + \frac{1}{30}\right)$$

$$f_g = -100\text{cm}$$

**Example: 3**

An equi convex lens of radii of curvature, 20cm is made of glass of refractive index 1.643 and 1.885 respectively for red and violet lights. If a small source of white light is placed on the principle axis of the lens at a distance of 35cm from the lens. Find the separation of the images formed by the red and violet constituents of light.

**Solution:**




$$\text{Seperation,} = V_R - V_V$$

But from;

$$\frac{1}{f} = (\cap - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right); \quad r_1 = r_2 = 20\text{cm}$$

For, Violet light; $\cap_V = 1.685$

$$\frac{1}{f_V} = (\cap_V - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

$$\frac{1}{f_V} = (1.685 - 1)\left(\frac{1}{20} + \frac{1}{20}\right) = 0.0685\text{cm}^{-1}$$

Then, using the lens formula;

$$\frac{1}{f_V} = \frac{1}{u_V} + \frac{1}{v_V}$$

$$0.0685 = \frac{1}{35} + \frac{1}{v_V}$$

$$v_V = 25\text{cm}$$

For, Red light; $\cap_R = 1.643$

$$\frac{1}{f_R} = (\cap_R - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

$$\frac{1}{f_R} = (1.643 - 1)\left(\frac{1}{20} + \frac{1}{20}\right) = 0.0643\text{cm}^{-1}$$

Then, using the lens formula;

$$\frac{1}{f_R} = \frac{1}{u_R} + \frac{1}{v_R}$$

$$0.0643 = \frac{1}{35} + \frac{1}{v_R}$$

$$v_R = 28\text{cm}$$

Thus the separation between the red and violet images is given by;

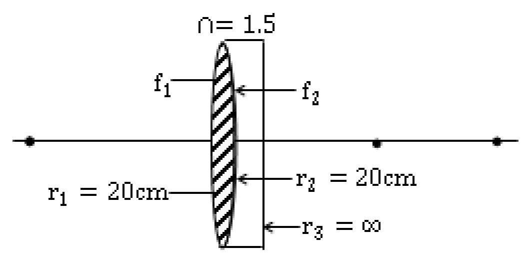
$$\text{Seperation,} = V_R - V_V$$
$$= 28 - 25$$
$$\text{Seperation,} = 3\text{cm}$$

**Example: 4**

A bi convex lens is placed in contact with a Plano concave lens. The radii of curvature of the surfaces of the biconvex lens are 20cm.The refractive index of the glass of the lenses is 1.5. Find the focal length of the combination,

**Solution:**

$$r_1 = r_2 = 20\text{cm and } r_3 = \infty$$




From;

$$\frac{1}{f_1} = (\cap - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right); \quad r_1 = r_2 = 20\text{cm}$$

$$\frac{1}{f_1} = (1.5 - 1)\left(\frac{1}{20} + \frac{1}{20}\right) = 0.05\text{ cm}^{-1}$$

Similarly;

$$\frac{1}{f_2} = (\cap - 1)\left(\frac{1}{r_2} + \frac{1}{r_3}\right); \quad r_2 = -20\text{cm}. r_3 = \infty$$

$$\frac{1}{f_2} = (1.5 - 1)\left(\frac{1}{-20} + \frac{1}{\infty}\right) = -0.025\text{ cm}^{-1}$$

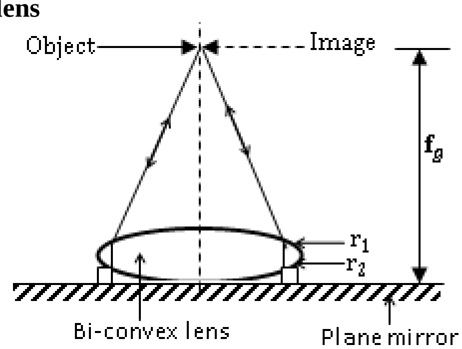
Then from the combined focal length;

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$$

$$\frac{1}{F} = 0.05 + -0.025 = 0.025$$

$$F = 40\text{cm}$$

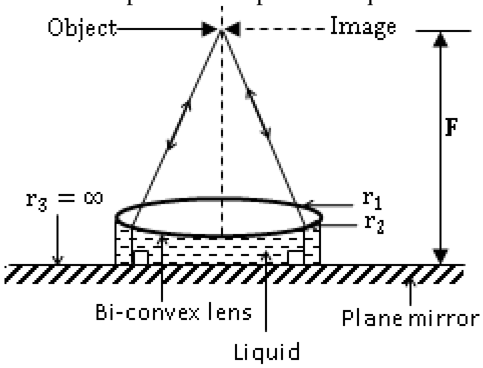
**Determination of the refractive index of a liquid using a convex lens**




A plane mirror is placed horizontally on a bench. A thin biconvex lens is placed on the mirror. The object pin is clamped horizontally in a retort stand so that it's pointed end lies along the axis of the lens.

The object pin is moved along the axis until the position is reached where the image of the pin coincides with the pin itself. The distance, $\mathbf{f_g}$of the pin from the plane mirror is measured and is the focal length of the lens.

The lens is then removed from the plane mirror. A small quantity of the specimen liquid is placed on the mirror; the bi convex lens is then placed on top of the liquid.




The object pin is moved along the axis until the new position is reached where the image of the pin coincides with the pin itself. The distance F of the pin from the plane mirror is measured and is the focal length of the liquid – glass lens combination.

Then, the focal length $f_L$ of the liquid lens can be obtained from; $\frac{1}{F} = \frac{1}{f_g} + \frac{1}{f_L}$:

Hence; the refractive index of the liquid is obtained from;

$$\frac{1}{f_L} = (\cap_L \quad - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

Where,

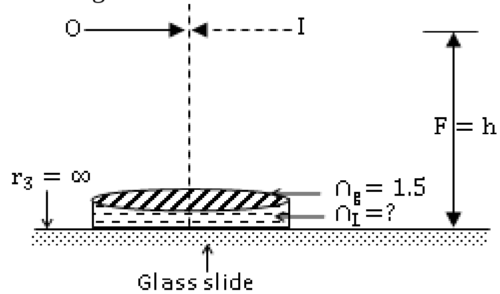
$r_1 = r =$ Radius of curvature of the glass lens

$r_2 = r_3 = \infty =$ Radius of curvature of the plane mirror

$$\frac{1}{f_L} = (\cap_L \quad - 1)\left(\frac{1}{r}\right) \Leftrightarrow \cap_L \quad = 1 + \frac{\mathbf{r}}{\mathbf{f_L}}$$

**Example: 1**

1. A converging lens is placed on top of a liquid of refractive index 1.4 and a glass slide as shown.




Using pin O, a position is found where O coincides with its image. If both surfaces of the lens have radii of curvature of 15cm and refractive index of the lens 1.5. Determine the distance h.

**Solution:**

For the liquid lens;

$r_1 = r = -15\text{cm}; r_2 = r_3 = \infty$

| | |
|---|---|
| $n_L = 1 + \frac{r}{f_L}$<br>$1.4 = 1 + \frac{-15}{f_L}$<br>$\frac{1}{f_L} = \frac{0.4}{15}$ | For the glass lens;<br>$\frac{1}{f_g} = (n_g - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$<br>$\frac{1}{f_g} = (1.5 - 1)\left(\frac{1}{15} + \frac{1}{15}\right) = \frac{1}{15}$ |

For the Glass-Liquid combination;

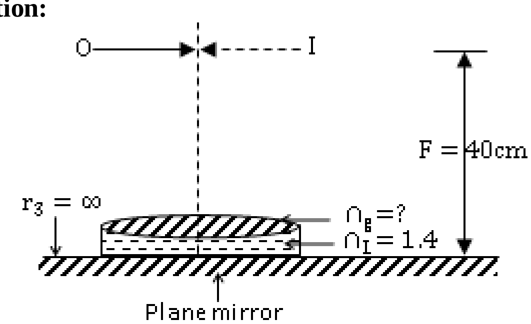
$$\frac{1}{h} = \frac{1}{f_1} + \frac{1}{f_g} \Leftrightarrow \frac{1}{h} = \frac{0.4}{-15} + \frac{1}{15} = \frac{1}{25}$$

$$h = 25\text{cm}$$

**Example: 2**

A bi-convex lens of radius of curvature 24cm is placed on a liquid film on a plane mirror. A pin clamped horizontally above the lens coincides with its image at a distance of 40cm above the lens. If the refractive index of the liquid is 1.4, what is the refractive index of the material of the lens?

**Solution:**




For a bi-convex lens;$r_1 = r_2 = 24\text{cm}$,

For the liquid lens, ; $r_1 = r_2 = -24\text{cm}$

[Negative because of the concave meniscus]

$$r_3 = \infty, \cap_L \quad = 1.4.$$

$\frac{1}{f_L} = (\cap_L \quad -1)\left(\frac{1}{r_1} + \frac{1}{r_3}\right); \quad r_1 = r_2 = -24\text{cm}$

$\frac{1}{f_L} = (1.4 - 1)\left(\frac{1}{-24} + \frac{1}{\infty}\right)$

[Negative sign is introduced because the upper side of the liquid is concave]

$\frac{1}{f_L} = (1.4 - 1)\left(\frac{1}{-24}\right)$

$f_L = -60\text{cm}$. This is the focal length of the liquid lens.

For the combined lenses, $F = 40\text{cm}$,

$\frac{1}{F} = \frac{1}{f_g} + \frac{1}{f_L}$

$\frac{1}{40} = \frac{1}{f_g} + \frac{1}{-60}$

$\frac{1}{f_g} = \frac{1}{40} + \frac{1}{60} = \frac{1}{24}$

$f_g = 24\text{cm}$

For glass lens, ; $r_1 = r_2 = 24\text{cm}, \cap_g \quad =?$

$$\frac{1}{f_g} = (\cap_g \quad -1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right); \quad r_1 = r_2 = 24\text{cm}$$

$$\frac{1}{24} = (\cap_g \quad -1)\left(\frac{1}{24} + \frac{1}{24}\right)$$

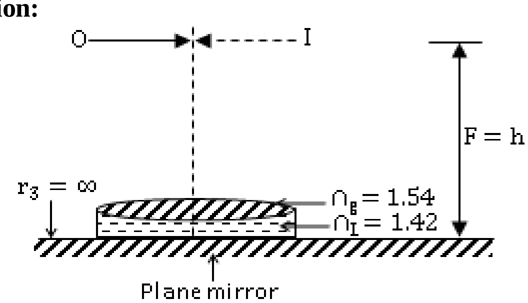
$$\cap_g = 1.5\text{cm}$$

This is the refractive index of the lens material.

**Example: 3**

An equi-convex lens of refractive index1.54 is placed on a horizontal plane mirror. A pin is moved along the horizontal axis of the lens and is found to coincide with its image at 25cm above the lens. The lens is then floated on the surface of mercury of refractive index 1.42 in a dish and the coincidence between the pin and its image is again obtained at a height, h above the lens. Find the value of h.

**Solution:**




For an equi − convex lens, ; $r_1 = r_2 = r$

$f_g = 25\text{cm}, F = h, \cap_g \quad = 1.54. \ \cap_L \quad = 1.42$

For the glass lens;

$$\frac{1}{f_g} = (\cap_g \quad -1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right); \quad r_1 = r_2 = r$$

$$\frac{1}{25} = (1.54 - 1)\left(\frac{1}{r} + \frac{1}{r}\right)$$

$$r = 27\text{cm}$$

For the Liquid lens;

$$\frac{1}{f_L} = (\cap_L \quad -1)\left(\frac{1}{r_2} + \frac{1}{r_3}\right); \ r_1 = r_2 = r, r_3 = \infty$$

$$\frac{1}{f_L} = (1.42 - 1)\left(\frac{1}{r} + \frac{1}{\infty}\right)$$

$$\frac{1}{f_L} = (1.42 - 1)\left(\frac{1}{r}\right)$$

$$\frac{1}{f_L} = (1.42 - 1)\left(\frac{1}{27}\right)$$

$$f_L = 64.3\text{cm}$$

For the combined lenses, $F = h$

$$\frac{1}{F} = \frac{1}{f_g} + \frac{1}{f_L}$$

$$\frac{1}{h} = \frac{1}{25} + \frac{1}{64.3} = \frac{393}{16075}$$

$$h = \frac{16075}{393} = 40.9\text{cm}$$

**Example: 4**

A Plano convex lens of refractive index 1.6 is placed on the surface of mercury in a dish with the plane surface in contact with mercury. It is found that the image of a pin coincides when the pin is 20cm from the lens. The lens is now inverted and floated with the curved surface in contact with mercury. Coincidence between the pin and its image is again obtained. Calculate the

(i). Focal length of the lens. [$f = 20\text{cm}$]

(ii). New position of the pin.. [$h = 7.5\text{cm}$]

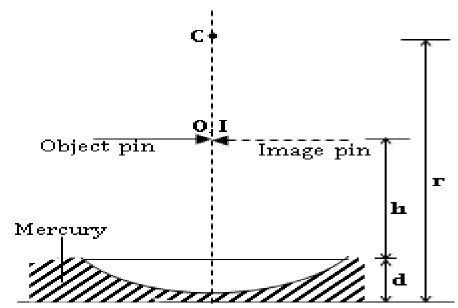
**Solution:**

Case I

When the Plano convex lens is floated with its plane surface in contact with mercury, the plane surface acts as a plane mirror and hence the distance of the point of coincidence from the lens represents the focal length of the lens.

Hence; $f_g = 20\text{cm}$

Case II:



But, d is very small; $d \approx 0$

$$\cap_g = \frac{\text{Real depth}}{\text{Apparent depth}} = \frac{r}{h + d} \approx \frac{r}{h}$$

$$h = \frac{r}{\cap_g}$$

$$\frac{1}{f_g} = (\cap_g - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right);\ r_1 = \infty;\ r_2 = -r$$

$$\frac{1}{20} = (1.6 - 1)\left(\frac{1}{\infty} + \frac{1}{-r}\right);$$

$$\frac{1}{20} = (1.6 - 1)\left(\frac{1}{-r}\right)$$

$$r = 1.2\text{cm}$$

Then from;

$$h = \frac{r}{\cap_g} \Leftrightarrow h = \frac{1.2}{1.6} \Leftrightarrow h = 7.5 \text{ cm}$$

**Exercise:**

1. Calculate the focal length of a converging meniscus with radii 25cm and 20cm with refractive index 1.5. [f=200cm]

2. A biconcave lens of radius of curvature 24cm is placed on a liquid film on a plane mirror. A pin clamped horizontally above the lens coincides with image at a distance of 40cm above the lens. If the refractive index of the liquid is 1.4, what is the refractive index of the material of the lens? (1.5)

3. A thin biconvex les is placed on a plane mirror. A pin is clamped above the lens so at it apex lies on the principal axial of the lens. The position of the pin is adjusted until the pin coincides with its image at a distance of 15cm from the mirror. When a thin layer of water of refractive index 1.33 is placed between the mirrors, the pin coincides with its image at a point 22.5cm from the mirror. When water is replaced by paraffin, the pin coincides with the image at a distance of 27.5cm from the mirror. Calculate the refractive index of paraffin. (1.45)

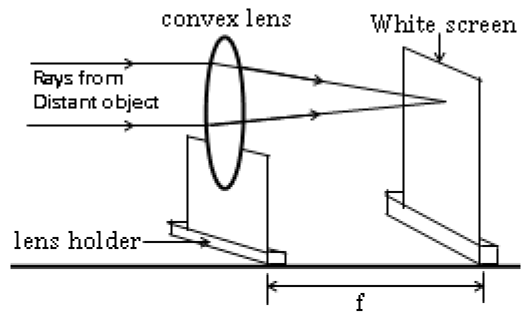
4. (a) A compound lens consists of two lenses in contact having powers of +12.5D and -2.5D. Find the position and nature of the image of an object placed 15cm from the compound lens. [30cm from the lens and it is real and magnified].

   (b) An equi convex lens of refractive index 1.5 is placed on a horizontal plane mirror and a pin held vertically above the lens is found to coincide with its image when positioned 20cm above the lens. When a few drops of a certain liquid are placed between the lens and the mirror, the pin has to be raised by 10cm to obtain coincidence with the image again. Find the refractive index of the liquid. [$\cap_L = \frac{4}{3} = 1.33$]

**Determining focal length of the converging lens (convex lens)**

**1. Rough method**

**(Using a distant object, e.g window)**




Position the lens and a white screen on a table as shown above.
Move the lens towards and away from the screen until a sharply focused image of the distant object is formed on the screen.
Measure the distance, f between the lens and the screen. It is approximately equal to the focal length of the lens used.

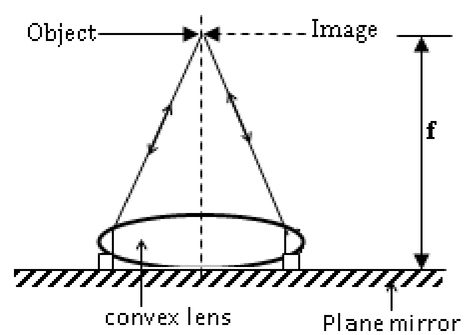
**Note:**
To improve the accuracy of the results, it is advisable that the experiment is repeated at least three times and the average focal length calculated.

| $f_1$(cm) | $f_2$(cm) | $f_3$(cm) | f(cm) |
|---|---|---|---|
| $a$ | $b$ | $c$ | $\frac{(a+b+c)}{3}$ |

**2. Plane mirror method**

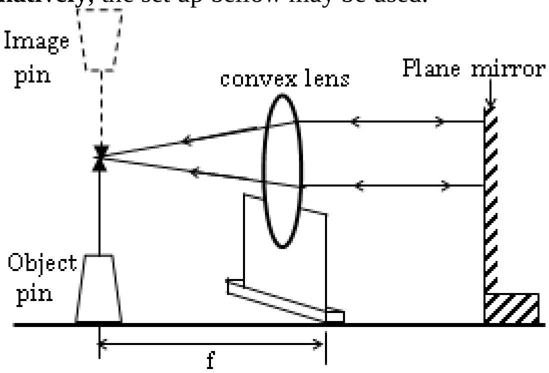
A plane mirror M is placed on a table with its reflecting surface facing upwards. The lens L is placed top of the mirror.

An optical pin, O is then moved along the axis of the lens until its image I coincides with the object O, when both are viewed from above and there is no parallax.



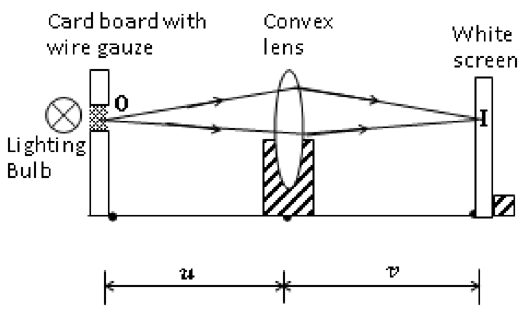

The distance from the pin O to the lens is thus measured and it is equal to the focal length, f, of the lens.

**Alternatively**, the set up bellow may be used.




**NOTE:** Rays from O passing through the lens are reflected from the plane mirror M and then pass through the lens a gain to form an image. When O and I coincide, the rays from O incident on the mirror must have returned a long their incident path after reflection from the mirror. This happens if the rays are incident normally on the plane mirror, M. The rays entering the lens after reflection are parallel and hence the point at which they converge must be the principle focus.

**3. Lens formula method**

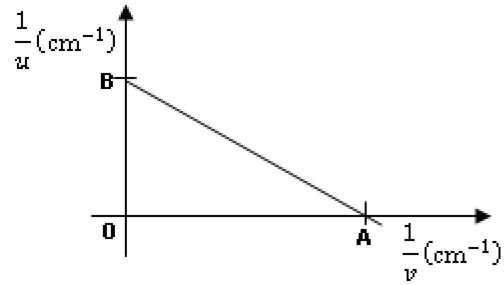



Using an illuminated object, O at a measured distance, u, move the screen towards and away from the lens until a clear image of the cross wires is obtained on the screen.

The image distance, $v$ is measured and recorded.
The procedure is repeated for various values of u and the corresponding values of v measured and recorded.
The results are tabulated including values of $\frac{1}{u}$ *and* $\frac{1}{v}$.

| $u$(cm) | $v$(cm) | $\frac{1}{u}(\text{cm}^{-1})$ | $\frac{1}{v}(\text{cm}^{-1})$ |
|---|---|---|---|
| - | - | - | - |

The focal length can be calculated from the equation $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$, and the average of the values obtained
Alternatively a graph of $\frac{1}{u}$ against $\frac{1}{v}$ is plotted and it gives a straight line graph through the points.




The intercepts A and B are read from the graph and focal length; $f$ is obtained from the equation;

$$f = \frac{1}{2}\left(\frac{1}{OA} + \frac{1}{OB}\right)$$

**Theory:**
From the lens formula; $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$

$$\Leftrightarrow \frac{1}{v} = (-1)\frac{1}{u} + \frac{1}{f}$$

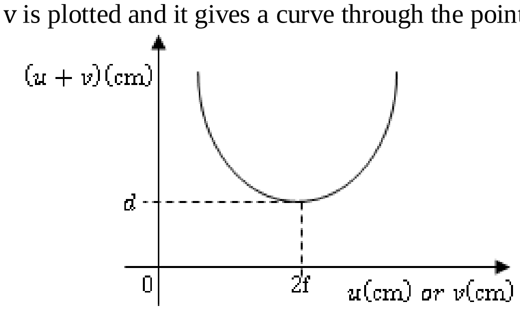
Compare with the general equation of a straight line,
$y = mx + c$

$$\Leftrightarrow m = -1, \quad \text{and} \quad c = \frac{1}{f} \Leftrightarrow f = \frac{1}{c}$$

Alternatively, the results are tabulated including values of $(u + v)$

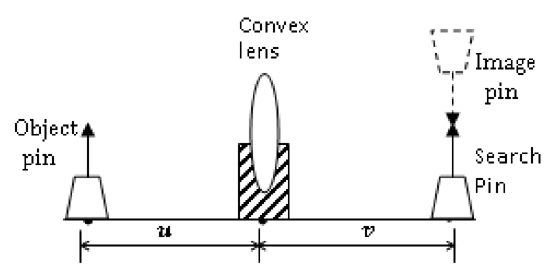
| $u$(cm) | $v$(cm) | $(u + v)$(cm) |
|---|---|---|
| - | - | - |

A graph of $(u + v)$ against $u$ or v is plotted and it gives a curve through the points.




The minimum value of the curve on the u or v- axis is read off and is equal to 2f.

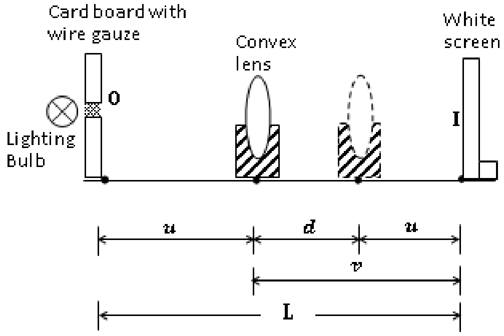
**Alternatively**, the set up bellow may be used.




An object pin is placed at a distance, u from the lens. A search pin is moved along the axis of the lens towards and away from the lens until its image, appears to coincide with the search pin.

The distance of this position from the lens is measured. It is the image distance, v.

### 4. Lens displacement method

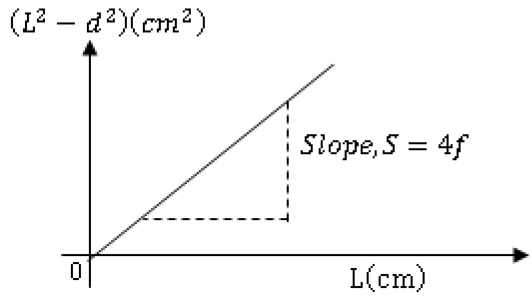



A convex lens whose focal length is to be determined is placed between an illuminated object and a screen.

The position of the screen is adjusted until a clear magnified image is obtained on the screen. The distance between the object and the screen, L is measured and recorded.

Keeping the screen fixed in its position, at a distance, **L** from the object, the lens is displaced and moved until a clear diminished image is obtained on the screen. The displacement, **d** through which the lens moves is measured and recorded.

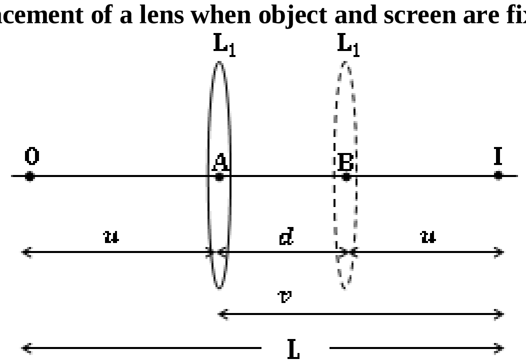
The results are tabulated including values of $L^2$-$d^2$
A graph of $L^2$-$d^2$ is plotted against L.




The focal length of the lens is thus measured from;

$$\mathbf{f} = \frac{1}{4}\boldsymbol{S}$$

**Displacement of a lens when object and screen are fixed.**




Object and image positions are fixed at O and I respectively.

The lens is moved in position $L_1$ when the image of O is produced at I, it's a gain displaced through a distance d until the image of O is a gain produced at I. Thus O and I are conjugate points.

OB = BI = u

$$2u + d = L \quad \Leftrightarrow \quad u = \frac{L-d}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

$$v = d + u$$

$$v = d + \frac{L-d}{2} \;\Leftrightarrow v = \frac{L+d}{2} \ldots\ldots\ldots\ldots\ldots\ldots. (ii)$$

Then, from the lens formula;

$$\frac{1}{\mathrm{f}} = \frac{1}{\mathrm{u}} + \frac{1}{\mathrm{v}}$$

$$\frac{1}{\mathrm{f}} = \frac{2}{\mathrm{L-d}} + \frac{2}{\mathrm{L+d}}$$

$$\frac{1}{\mathrm{f}} = \frac{2(\mathrm{L+d}) + 2(\mathrm{L-d})}{(\mathrm{L-d})(\mathrm{L+d})}$$

$$\frac{1}{\mathrm{f}} = \frac{4\mathrm{L}}{\mathrm{L}^2 - \mathrm{d}^2}$$

$$\mathbf{f} = \frac{\mathbf{L}^2 - \mathbf{d}^2}{\mathbf{4L}}$$

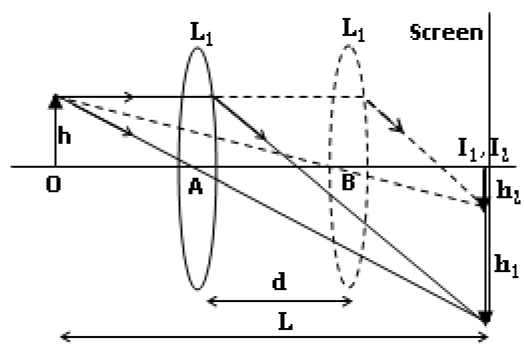
Thus since, L and ***d*** are measurable distances, then the focal length, f of the lens can be calculated.

**Theory:**

$$\text{From, } \mathrm{f} = \frac{\mathrm{L}^2 - \mathrm{d}^2}{4\mathrm{L}} \quad \Leftrightarrow \mathrm{L}^2 - \mathrm{d}^2 = (4f)L$$

Compare with; $\quad y = mx + c$

$$\Leftrightarrow \mathrm{m = slope, S = 4f \;\; and \;\; c = 0}$$




When the lens is in position A,

$$m_1 = \frac{v}{u} = \frac{AI}{OA} = \frac{\left(\frac{L+d}{2}\right)}{\left(\frac{L-d}{2}\right)} = \frac{L+d}{L-d}$$

When the lens is in position B,

$$m_2 = \frac{v}{u} = \frac{BI}{OB} = \frac{\left(\frac{L-d}{2}\right)}{\left(\frac{L+d}{2}\right)} = \frac{L-d}{L+d}$$

Thus the effective magnification, M is given by;

$$M = m_1 m_2$$

$$M = \left(\frac{L+d}{L-d}\right) \times \left(\frac{L-d}{L+d}\right) = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

$$\text{But, } m_1 = \frac{h_1}{h}; \quad \text{and} \quad m_2 = \frac{h_2}{h}$$

Where h is the height of the object, $h_1$ and $h_2$ are the heights of the image when the lens is at A and B respectively.
Thus the effective magnification, M is given by;
$M = m_1 m_2$

$$M = \frac{h_1}{h} \times \frac{h_2}{h} = \frac{h_1 h_2}{h^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

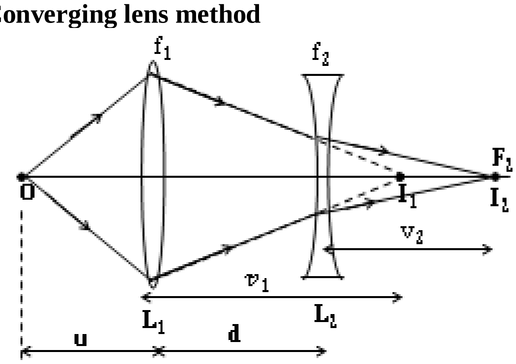
From equations (i) and (ii), $M = 1$

$$1 = \frac{h_1 h_2}{h^2}$$
$$h^2 = h_1 h_2$$
$$\mathbf{h = \sqrt{h_1 h_2}}$$

The length h of the object can be found by measuring the heights h1 and h2 of the images for the two positions of the lens at A and B respectively.
This method of f and h is most useful when the object and lens are inaccessible. E.g when the lens is in a tube and measurements of u and v **cannot** be made. (Spectrometer).

**Determining focal length of the diverging lens (concave lens)**

**1. Converging lens method**




An object, O is placed at a distance, u from the convex lens $L_1$ greater than its focal length, so that it forms an image $I_1$. Measure the distance, $\boldsymbol{v_1}$.

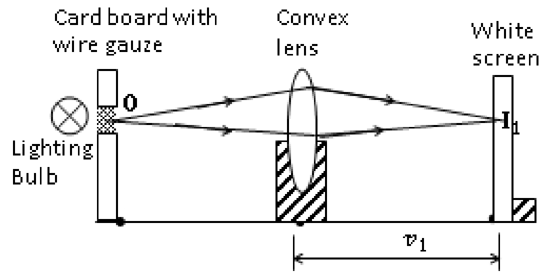
Lens $L_2$ is then placed between $L_1$and $I_1$. $I_1$ acts as a virtual object for the diverging lens, $L_2$. and a real image $I_2$ is formed by $L_2$. Measure the distances, $\boldsymbol{d}$ and$\boldsymbol{v_2}$.

The object distance, $u_2$ from the diverging lens to $I_1$ is; $u_2 = -(v_1 - d)$ and $d$ can be measured. Its negative, since $I_1$is virtual for the concave lens so, from $\frac{1}{f} = \frac{1}{u_2} + \frac{1}{v_2}$ , the focal length f is calculated.

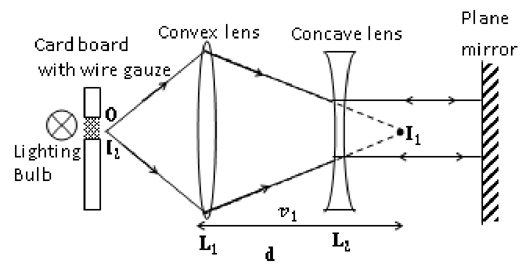
**NOTE:**
By itself, a diverging lens always forms a virtual image of a real object. A real image may be obtained, however, if a virtual object is used, and a converging lens can provide such an object.

$I_2$ is further a way from $L_2$and $I_1$, since the concave lens makes the incident beam on it diverge more. The distance of the final image, $v_2$ , from the diverging lens is measured; It is positive in sign as $I_2$ is real.

**Alternatively: By using a converging lens, plane mirror and screen.**



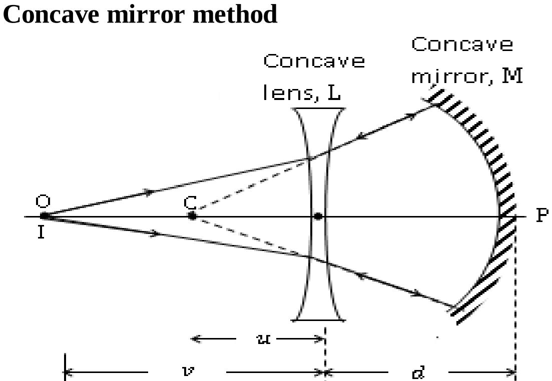

An illuminated object is placed behind a slit. Using a convex lens, focus a sharp image, $I_1$onto a screen.
Measure and record the distance, $v_1$ between the convex lens and the screen.




A concave lens whose focal length is required is then placed between the convex lens and the screen.
Remove the screen and re place it with a plane mirror. Adjust the position of the concave lens until a point is reached when the final image coincides with the object.
Measure and record the distance, $d$ between the convex lens and the concave lens.
The focal length of the concave lens is then determined from;

$$\mathbf{f = v_1 - d}$$

**2. Concave mirror method**

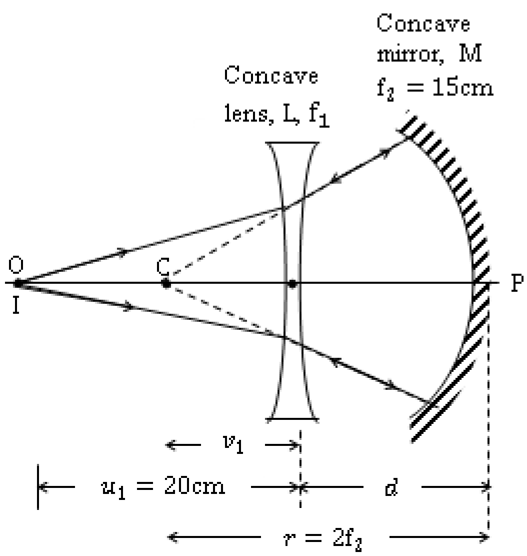


An illuminated object, O is placed beyond the centre of curvature of a concave mirror of known radius of curvature, r.
A diverging length, L is then placed between I and M. The position of L is then adjusted until a clear image, I is formed besides O.
Distances, $\mathrm{LI} = v = r$, and $\mathrm{LM} = \mathrm{d}$ are measured and recorded.
For the concave lens; $u = -(\mathrm{CL}) = -(\mathrm{r} - \mathrm{LM})$

$v = (\mathrm{IL})$

The focal length, f of the concave lens is then calculated from, $\frac{1}{\mathrm{f}} = \frac{1}{\mathrm{u}} + \frac{1}{\mathrm{v}}$

**Example 1:[UNEB 2002 Qn.1 (b)]**
An object is placed 20cm in front of a diverging lens placed coaxially with a concave mirror of focal length 15cm. When the concave mirror is 20cm from the lens, the final image coincides with the object.
(i) Draw a ray diagram to show the formation of the final image.
(ii) Determine the focal length of the diverging lens.
**Solution:**
**(i)**




**(ii)**
Considering action of the concave lens
$u_1$=20cm, $v_1$=-(r - 20) $r = 2f_2$
=-(30-20) $r = 2 \times 15$
=-10cm r = 30cm
Using the lens formula;

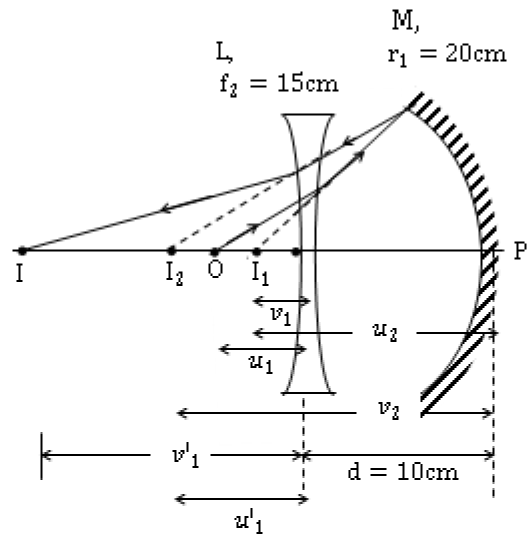
$$\frac{1}{f_1} = \frac{1}{u_1} + \frac{1}{v_1}$$

$$\frac{1}{f_1} = \frac{1}{20} + \frac{1}{-10}$$

$$f_1 = -20cm$$

Hence, the focal length of diverging lens is 20cm

**Example 2:**
A concave mirror of radius of curvature 20cm is arranged coaxially with a concave lens of focal length 15cm, placed 10cm from the mirror. An object 3cm tall is placed in front of a concave lens and its image is formed on a screen 40cm away from the lens.
(i) Find the position of the object.
(ii) What is the height of the image formed?
(iii) Explain what would happen if the lens was replaced with similar lens but of a much smaller focal length.
**Solution:**




| | |
|---|---|
| $V_1$ =40cm<br>$U_1$ =? $F_1$=-15cm (diverging lens)<br>Using, $\frac{1}{f_1} = \frac{1}{u_1} + \frac{1}{v_1}$<br>$\frac{1}{u_1} = \frac{1}{-15} - \frac{1}{-40}$<br>$u_1 = -10.9cm$<br>For the concave mirror<br>$v_2 = u_1 + 10 = 20.9cm$<br>$u_2 = v_1 + 10$<br>$f_2 = {}^{r}/_{2} = {}^{20}/_{2} = 10cm$<br>Using $\frac{1}{f_2} = \frac{1}{u_2} + \frac{1}{v_2}$<br>$\frac{1}{10} = \frac{1}{v_1 + 10} + \frac{1}{20.9}$<br>$\frac{1}{v_1 + 10} = \frac{1}{10} - \frac{1}{20.9}$<br>$v_1 + 10 = 19.17$<br>$v_1 = 9.17cm$ | For the concave lens;<br>$v_1 = -9.17\ cm.(virtual)$<br>$f_1 = -15cm$<br>$u_1$ =?<br>Using $\frac{1}{f_1} = \frac{1}{v_1} + \frac{1}{u_1}$<br>$\frac{1}{-15} = \frac{1}{u_1} + \frac{1}{-9.17}$<br>$\frac{1}{u_1} = \frac{1}{9.17} - \frac{1}{15}$<br>$u_1 = 23.59cm$<br>The object is 23.59cm in front of the concave lens. |

**ii).** $\boldsymbol{h}_1 = \frac{v_1}{u_1} \times \boldsymbol{h}_0$

$$\boldsymbol{h}_1^1 = \frac{\boldsymbol{v}_2}{\boldsymbol{u}_2} \times \boldsymbol{h}_1$$

$$h'_1 = \frac{v_2}{u_2} \times \frac{v_1}{u_1} \times h_0$$

$$h''_1 = \frac{v'_1}{u'_1} \times h'_1$$

$$h''_1 = \frac{v'_1}{u'_1} \times \frac{v_2}{u_2} \times \frac{v_1}{u_1} \times h_0$$

$$h''_1 = \frac{40}{10.9} \times \frac{20.9}{19.17} \times \frac{9.17}{23.59} \times 3$$

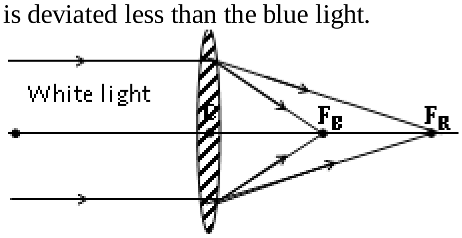
$$h''_1 = 4.67\text{m}$$

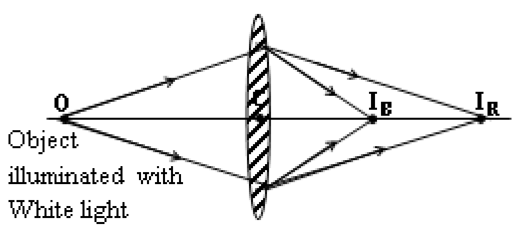
iii). A lens of a shorter focal length has a higher power implying that it would diverge the light more. Therefore, since the positions of the object, lens, and mirror are not changed, the final image will be formed further away from the lens than even before and it will be bigger in size

**Defects of image formed by lenses**

1. **Chromatic aberration**

This is when a wide beam of light incident on a lens produces coloured images.
This happens because the lens has different focal lengths for the different colours of white light. As a result, images corresponding to the different colours are formed in different locations along the principal axis of the lens.
Red light is deviated less than the blue light.

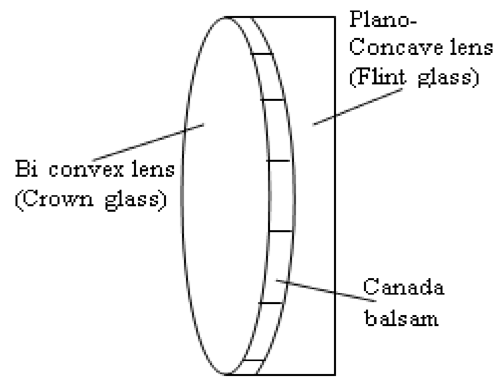



This causes the image of an object illuminated by white light to be blurred with coloured edges.

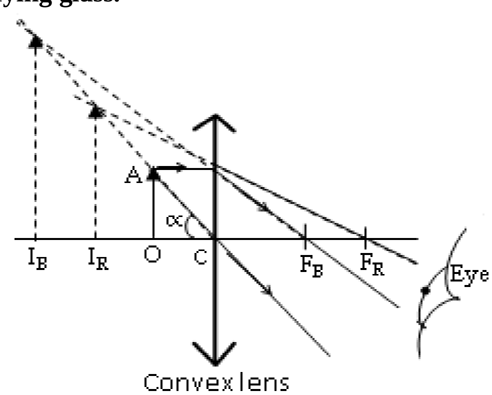
***Minimizing chromatic aberration***

(i). Chromatic aberration can be reduced by using <u>achromatic doublet</u>. This consists of a bi-convex lens just in contact with a Plano concave lens different refractive indices, using Canada balsam.
The convex lens deviates the rays while the concave nullifies the deviations.




(ii). By keeping the eye as close as possible to the lens when the images formed in it.

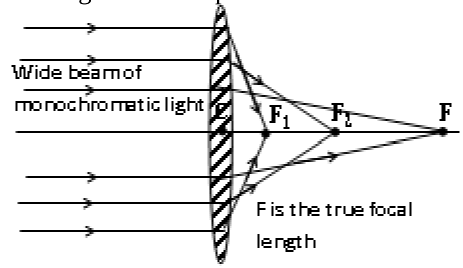
**Why chromatic aberration is not experienced in a magnifying glass.**

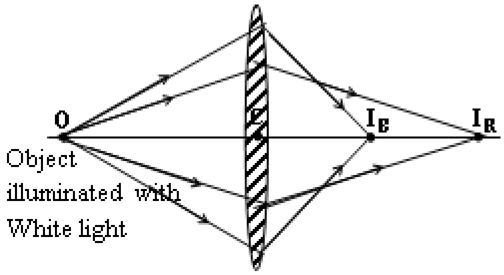



It is because various images of the object, OA are formed and the tip of each lies along the line CA but each image subtends the same angle at the eye. Thus the images appear superimposed and hence the defect is not noticed.
The colours received by the eye overlap and the virtual images seen are free from chromatic aberration.
Note: Chromatic aberration is a defect in lenses only.

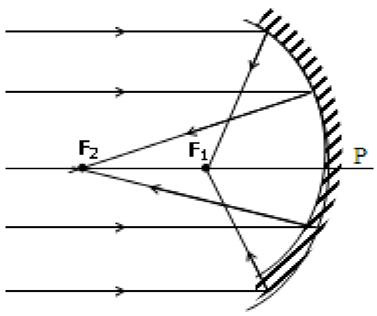
**2. Spherical aberration.**

This happens in both mirrors and lenses**.**
Consider a wide beam of monochromatic light incident on a wide aperture converging lens. Marginal rays (far away from the principal focus) converge near the lens than the central rays.
Consequently, the image formed is blurred (distorted). This defect of the image is called spherical aberration.





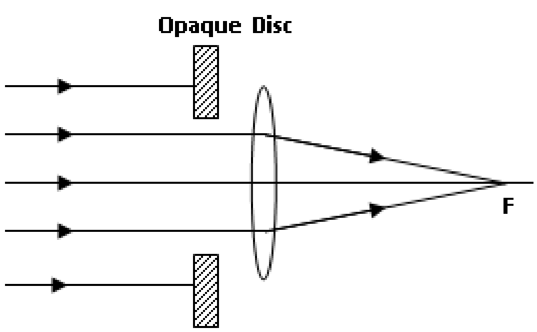

With a lens of a small aperture, all rays incident on it are focused at the same point forming a sharp image of a clearly defined object.
This defect also occurs with wide aperture spherical mirror.




***Minimizing spherical aberration***

(i). Grinding the lens:
The purpose of grinding the lens is to reduce the aperture of the lens, so that marginal rays incident on the lens are cut off.
Lenses of small aperture do not suffer from spherical aberration; this explains why they are preferred to those of wider apertures.

(ii). Stopping the lens:
Spherical aberration can be minimized using a stopper i.e. using an opaque disc with a central hole.

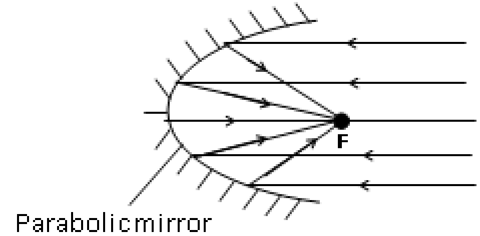



The disc will stop marginal rays from reaching the lens. The disadvantage with this method is light intensity is cut down and so the brightness of the image is reduced.

Note: Using a disc has a disadvantage that the amount of light energy passing through te lens reduces and consequently the brightness of the images reduces.

(iii). By making all the angles of incident from the object to the lens as small as possible. This is done by using a Plano convex or Plano concave lens.

(iv). using a parabolic mirror instead of a concave mirror.
In mirrors, spherical aberration can be minimized by using a parabolic mirror instead of a concave mirror.




This brings all rays parallel to principal axis to converge at the same focus.

**Exercise**

1. An illuminated object is placed on the 0cm mark of an optical bench. A converging lens of focal length 15cm is placed at the 22.5cm mark. A diverging lens of focal length 30cm and a plane mirror are placed at the 37.5cm and 77.5cm marks respectively. Find the position of the final image. (at 0cm mark). Illustrate your answer with a ray diagram.

2. A converging lens of focal length 1.5cm is placed 29.0cm in front of another converging lens of 6. 25cm.An object of height 0.1cm is placed 1.6cm away from the first lens on the side remote from the second lens at right angles to the principal axis of the final image by the system. Determine the position and size and final image of the object.

3. An equi-convex lens A is made of glass of refractive index 1.5 and has a power of 5.0radm$^{-1}$. It is combined in contact with a lens B to produce a combination whose power is 1.0radm$^{-1}$. the surfaces in contact fits exactly. The refractive index of the glass in lens B is 1.6. What are the radii of the four surfaces? Draw a diagram to illustrate your answer.

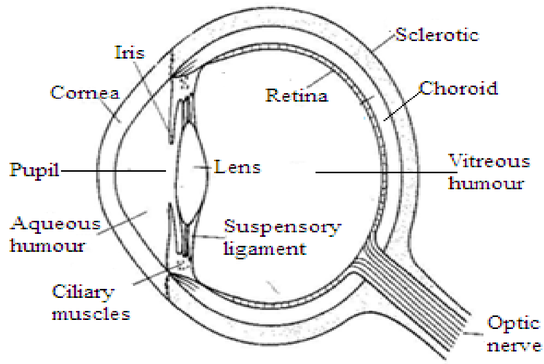
4. A lens forms the image of a distant object on a screen 30cm away. Where a second lens of focal length 30cm should be placed so that the screen has to be moved 8cm towards the first lens for the new image to be in focus?

5. A convex lens of focal length 20cm forms an image on a screen placed 40cm beyond the lens. A concave lens of focal length 40cm is then placed between a convex lens and a screen a distance of 20cm from the convex lens.
   (i) Where must the screen be placed in order to receive the new image see.
   (ii) What is the magnification produced by the lens system?

# OPTICAL INSTRUMENTS.

An optical instrument is a device that aids vision. They operate in presence of light.

## THE EYE




**Functions of the parts of the eye.**

**Lens;** The lens inside the eye is convex. It's sharp; it changes in order to focus light.

**Ciliary muscle**: These alter the focal length of lens by changing its shape so that the eye can focus on image on the retina.

**The iris**: This is the coloured position of the eye. It controls the amount of light entering the eye by regulating the size of the pupil

**The retina:** This is a light sensitive layer at the back of the eye where the image is formed.

**The optic nerve**: It is the nerve that transmits the image on the retina to the brain for interpretation.

**The cornea:** It is the protective layer and it also partly focuses light entering the eye

**ACCOMMODATION**

This is the process by which the human eye changes its size so as to focus the image on the retina. This process makes the eye to see both near and far objects.

**Near point:**

It's the nearest position at which an object is seen most clearly. The distance from the eye to the near point is called ***Least Distance of Distinct Vision***, **D**. For a normal eye, $\mathbf{D} = 25$**cm.**

An object whose distance from the eye is less than D appears blurred, while the one which is further away appears smaller than when it is at the near point.

When viewing an object, the near point, the eye is said to be fully accommodated.

**Far point:**

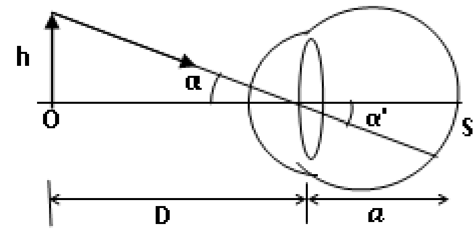
It's the furthest position at which an object is seen most clearly.

For a normal eye, the furthest distance is **infinity**.

The normal eye can focus an object at infinity (far point of a normal eye), and in this case, the eye is said to be un accommodated.

**Virtual angle:**

Angle subtended at the un aided eye by the object. or it the angle subtended at the eye by the image when instrument is being used.




$\propto = Visual\ angle.$

$Arc\ length, S = Sise\ of\ the\ image.$

$$S = a \propto$$

Where **a** is the distance between the eye lens and the retina.

Thus, Size of the image formed at the retina is directly proportional to the visual angle.

***Explain why the further vertical pole in line with others of equal height looks shorter.***

The image size depends on the visual angle. The farthest pole subtends a small visual angle. Hence Image size is smaller

**Angular magnification (magnifying power).**

It is the ratio of the angle subtended at the eye by the image when instrument is being used to the angle subtended at an unaided eye by the object.

i.e. angular magnification, $\mathbf{M} = \frac{\alpha^1}{\alpha}$

Where **α'** is angle subtended at the eye by the image when instrument is used and **α** is the angle subtended at an unaided eye by the object.

**EYE DEFECTS AND THEIR CORRECTIONS**

The normal eye can see objects clearly placed at infinity ( far point) to see objects in greater details the eye sees it at the near point i.e 25cm

TYPES OF EYE DEFECTS

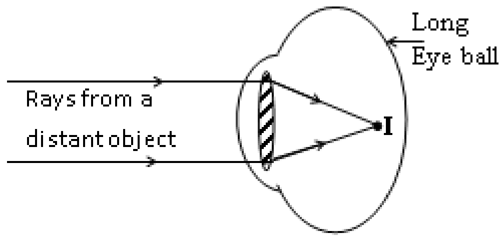
a) Short sightedness
b) Long sightedness

SHORT SIGHTEDNESS

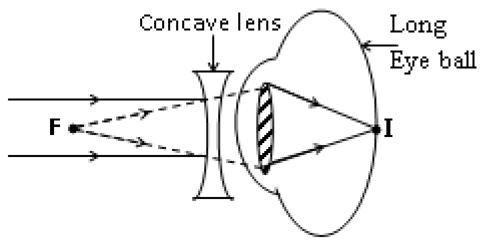
A person with short sightedness can see near objects clearly but distant objects are blurred. The furthest point at which one can see the objects clearly is the far point. An object which is further than the far point is focused in front of the retina.

It is caused by;

- Very long eyeball
- Lens of very short focal length



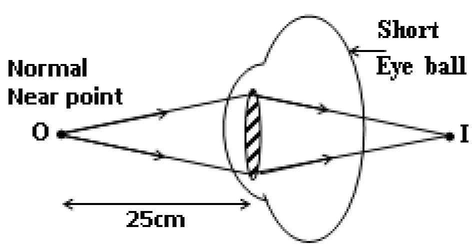

<u>Correction of short-sightedness</u>



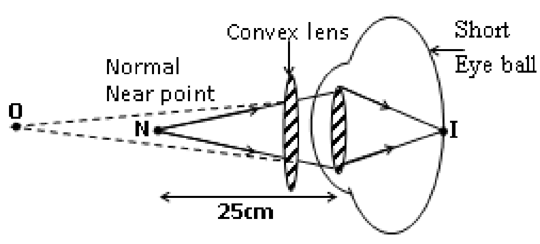

A concave lens is placed in front of the eye to make the light diverge so that it appears to come from the near point when it's actually coming far away as shown above.

<u>LONGSIGHTEDNESS</u>
A long sighted person can see distant objects clearly but those that are near are blurred. The nearest point at which the person can see an object clearly is called near point. An object placed near than the near point is focused behind the retina as shown below.
It is caused by;

- Very short eyeball
- Lens of very long focal length



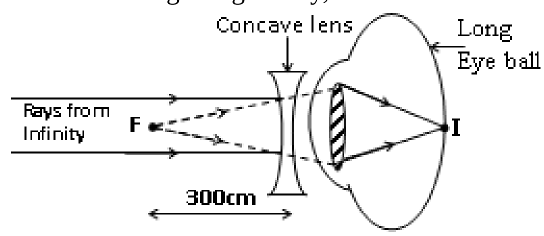

<u>Correction of long sightedness:</u>




A convex lens is placed in front of the eye to make the light parallel, so that it appears to come from a distant object as shown above.

**Examples:**
1. A person can focus objects when they lie between50cm and 300cm from his eyes. Finding the range of distinct vision when using each pair of spectacles, what spectacles should he use in order to;
(i) Increase his maximum distance of distinct vision to infinity.
(ii) Reduce his least distance of distinct vision to 25cm.

**Solution:**
(i) In order to Increase his maximum distance of distinct vision from 300cm to infinity, the person requires a diverging lens whose focal length is given by;




$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

$$\frac{1}{f} = \frac{1}{\infty} + \frac{1}{-300} \Leftrightarrow \frac{1}{f} = \frac{1}{-300} \Leftrightarrow f = -300\text{cm}$$

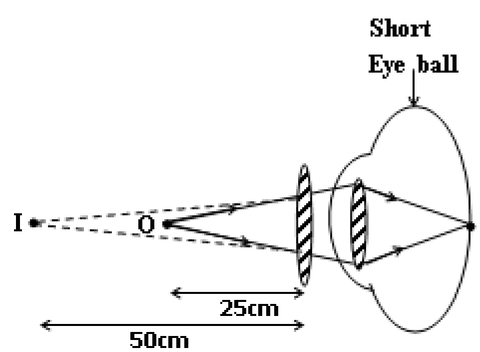
Thus from the above, we note that when using these spectacles, one limit of distinct vision becomes infinity and the other limit of the range is given by using a lens formula in order to determine where the object should be placed in order to appear to be at 50cm.

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

$$\frac{1}{-300} = \frac{1}{u} + \frac{1}{-50} \Leftrightarrow \frac{1}{u} = \frac{1}{60} \Leftrightarrow u = 60\text{cm}$$

<u>Thus the range of distinct vision when using these spectacles becomes 60cm to infinity.</u>

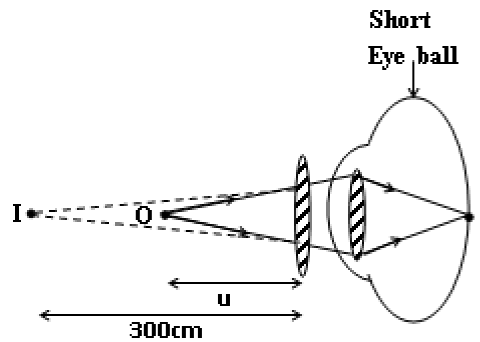
(ii) In order to reduce his least distance of distinct vision to 25cm. The person requires a converging lens. Then in this case, if we assume that the lens is close to the eye, then the lenses will make objects at a distance of 25cm to appear as if they are at 50cm.



$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

$$\frac{1}{f} = \frac{1}{25} + \frac{1}{-50} \Leftrightarrow \frac{1}{f} = \frac{1}{50} \Leftrightarrow f = 50\text{cm}$$

When using this type of spectacles, his maximum distance of distinct vision will be given by;




$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

$$\frac{1}{50} = \frac{1}{u} + \frac{1}{-300} \Leftrightarrow \frac{1}{u} = \frac{1}{50} + \frac{1}{300}$$

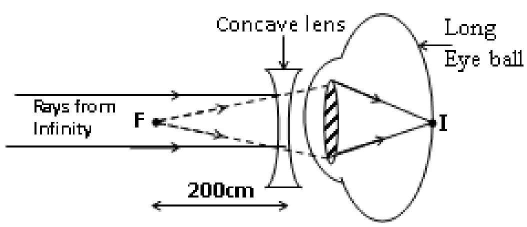
$$u = 42.86\text{cm}$$

<u>Thus the range of distinct vision when using these spectacles becomes 25cm to 42.86.</u>

2. A certain person can see clearly objects between 20cm and 200cm from his eyes. What spectacles are required to enable him see distant objects clearly. What will be his least distance of distinct vision when using those spectacles?

**Solution:**

In order, for this person to be able to see distant objects he should put on spectacles containing diverging lenses since his furthest point is 200cm from the eye, the spectacles should have a diverging lens of focal length, f = 200cm such that rays from distant objects appear to come from the focal point.

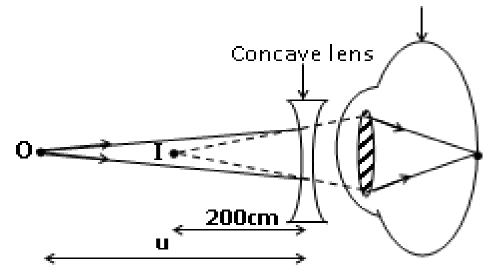



$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

$$\frac{1}{f} = \frac{1}{\infty} + \frac{1}{-200} \Leftrightarrow \frac{1}{f} = \frac{1}{-200} \Leftrightarrow f = -200\text{cm}$$

He should use diverging lens of f = 200cm.

When using these spectacles, his nearest or closest distance of distinct vision is going to be the point at which the object should be put so that it appears to come from a point 20cm from the eye.




$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

$$\frac{1}{-200} = \frac{1}{u} + \frac{1}{-20} \Leftrightarrow \frac{1}{u} = \frac{1}{-200} + \frac{1}{20}$$

$$u = 22.22\text{cm}$$

Thus his least distance of distinct vision is 22.22cm. His range of clear vision is 22.22cm to infinity.

## MICROSCOPES

These are used to view near objects when in normal use. The eye is accommodated. The image formed is at the least distance of distinct vision from the eye.

**Angular magnification of microscopes** $= \frac{\alpha^1}{\alpha}$

**α** is the angle subtended at the eye by an object at the near point when microscope is not used.

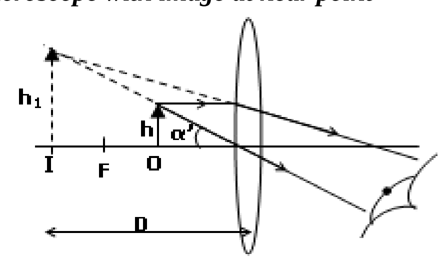
**α'** is the angle subtended at the eye by the image when a microscope is used.

<u>In normal adjustment or use, the microscope forms the image at the near point.</u>

### (a) Simple microscope / magnifying glass

This consists of a single convex lens with the distance between the object and the lens less than or equal to the focal length of the lens.

***Simple microscope with image at near point***

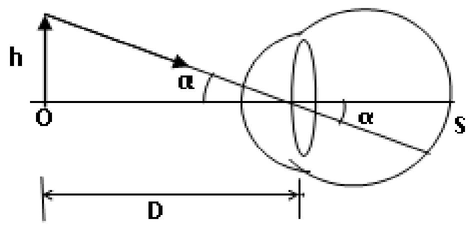



$$m = \frac{\alpha^1}{\alpha}$$

For small angles in radians

$$\alpha' \approx \tan\alpha' = \frac{h_1}{D}$$

If $\alpha$ is the angle subtended at the eye by the object at the near point.




$\alpha \approx \tan\alpha = \dfrac{h}{D}$

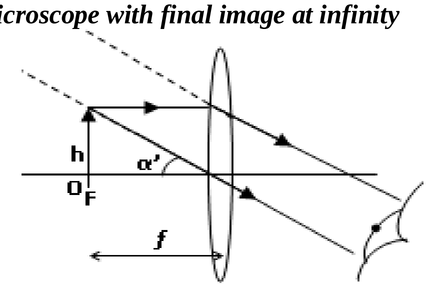
Hence $m = \frac{\alpha^1}{\alpha} = \frac{\frac{h_1}{D}}{\frac{h}{D}} = \frac{h_1}{h}$

But $\frac{h_1}{h}$ is the linear magnification.

Hence, $\frac{h_1}{h} = \frac{D}{f} - 1$

Therefore angular magnification, $m = \frac{D}{f} - 1$

***Simple microscope with final image at infinity***




Angular magnification, $\mathrm{M} = \frac{\alpha^1}{\alpha}$

$\alpha' \tan\alpha' = \dfrac{\mathrm{h}}{\mathrm{f}}$

$\alpha \approx \tan\alpha = \dfrac{\mathrm{h}}{\mathrm{D}}$

Hence angular magnification**,**

$$\mathrm{M} = \frac{\alpha^1}{\alpha} = \frac{\left(\frac{\mathrm{h}}{\mathrm{f}}\right)}{\left(\frac{\mathrm{h}}{\mathrm{D}}\right)} = \frac{\mathrm{D}}{\mathrm{f}}$$

$$\mathrm{m} = \frac{\alpha'}{\alpha} = \frac{\left(\frac{\mathrm{h}_2}{\mathrm{D}}\right)}{\frac{\mathrm{h}}{\mathrm{D}}} = \frac{\mathrm{h}_2}{\mathrm{h}}$$

$$\mathbf{m} = \left(\frac{\mathbf{h}_1`}{\mathbf{h}}\right) \times \left(\frac{\mathbf{h}_2}{\mathbf{h}_1}\right) = \left(\frac{\mathbf{v}_0}{\mathbf{f}_0} - 1\right) \times \left(\frac{\mathbf{D}}{\mathbf{f}_\mathbf{e}} - 1\right)$$

**Note:**

(i) Angular magnification is higher when a simple microscope forms the image at the near point.

(ii)For higher magnification, use lenses of short focal length.

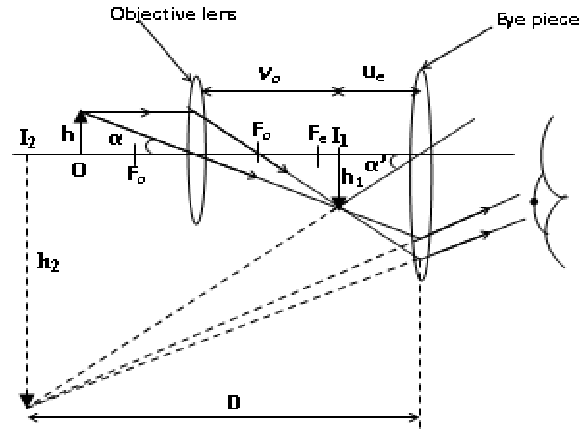
**(b) Compound microscopes**

$f$ cannot be decreased indefinitely to achieve high angular magnification. This requires the use of two converging lenses, namely the objective (which is near the object) and the eye piece, near the eye.

The focal length of each lens should be small such that a high overall angular magnification is produced.

***Compound microscope in normal adjustment*.**

In normal adjustment, a compound microscope forms the image at the least distance of distinct vision from the eye i.e. image at near point.




The objective lens forms an intermediate real image $I_1$, of the object O. $I_1$ is formed at a point nearer the eye piece than the principal focus $f_e$ of the eye piece.

The eye piece acts as a magnifying glass. It forms a virtual image $I_2$ of $I_1$. The observer's eyes should be taken to be close to the eye piece so that α' is the angle subtended at the eye by the final virtual image $I_2$.

For the aided eye, and for small angles in radians,

$\alpha' \approx \tan\alpha' = \dfrac{\mathrm{h}_2}{\mathrm{D}}$

For an unaided eye, and for small angles in radians,

$$\alpha \approx \tan\alpha = \frac{\mathrm{h}}{\mathrm{D}}$$

Angular magnification, $\mathrm{M} = \frac{\alpha^1}{\alpha}$

$$\mathrm{M} = \frac{\alpha^1}{\alpha} = \frac{\mathrm{h}_2}{\mathrm{D}} \div \frac{\mathrm{h}}{\mathrm{D}} = \frac{\mathrm{h}_2}{\mathrm{D}} \times \frac{\mathrm{D}}{\mathrm{h}} = \frac{\mathrm{h}_2}{\mathrm{h}}$$

$$\mathrm{M} = \frac{\mathrm{h}_2}{\mathrm{h}}$$

$\frac{\mathrm{h}_2}{\mathrm{h}}$ is by definition the lateral magnification, m.

For a magnifying glass with final image at near point,

$$\mathrm{M} = \mathrm{m} = \frac{\mathrm{h}_2}{\mathrm{h}}$$

$$\mathrm{M} = \frac{\mathrm{h}_2}{\mathrm{h}} = \frac{\mathrm{h}_2}{\mathrm{h}_1} \times \frac{\mathrm{h}_1}{\mathrm{h}}$$

$$\mathbf{M} = \frac{\mathbf{h}_2}{\mathbf{h}} = \mathbf{M}_\mathbf{e} \times \mathbf{M}_\mathbf{o}$$

Where $\mathbf{M}_\mathbf{e}$ and $\mathbf{M}_\mathbf{o}$are the linear magnifications produced by the eyepiece and objective respectively.

$$\mathbf{M}_\mathbf{e} = \frac{h_2}{h_1} = \left(\frac{D}{f_e} - 1\right);\ \mathbf{M}_\mathbf{o} = \frac{h_1}{h} = \left(\frac{v_0}{f_0} - 1\right)$$

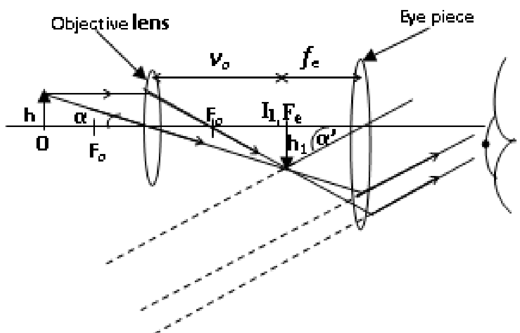
$$\mathrm{M} = \mathrm{M}_\mathrm{e} \times \mathrm{M}_\mathrm{o}$$

$$\mathbf{M} = \left(\frac{\mathbf{D}}{\mathbf{f}_\mathbf{e}} - 1\right)\left(\frac{\mathbf{v}_\mathbf{o}}{\mathbf{f}_\mathbf{o}} - 1\right)$$

**Note:** For higher angular magnification, both the eye piece and the objective should have short focal lengths.

***Compound microscope with final image at infinity***
**(Not in normal adjustment)**

The object is outside $F_0$.




The separation of the objective and the eye piece is such that the object forms an intermediate image of the object at the principle focus, $F_e$ of the eye piece; hence the eye piece focuses the final image at infinity.

The angle α' subtended by the final image by the eye piece is

For the aided eye, and for small angles in radians,

$\alpha' \approx \tan\alpha' = \frac{h_1}{f_e}$

For an unaided eye, and for small angles in radians,

$$\alpha \approx \tan\alpha = \frac{h}{D}$$

Angular magnification, $M = \frac{\alpha^1}{\alpha}$

$$M = \frac{\alpha^1}{\alpha} = \frac{h_1}{f_e} \div \frac{h}{D} = \frac{h_1}{h} \times \frac{D}{f_e}$$

But $\frac{h_1}{h} = \left(\frac{v_0}{f_0} - 1\right)$ is the linear magnification produced by the objective lens.

$$\boldsymbol{M} = \left(\frac{\boldsymbol{v}_0}{\boldsymbol{f}_0} - 1\right)\left(\frac{\boldsymbol{D}}{\boldsymbol{f}_e}\right)$$

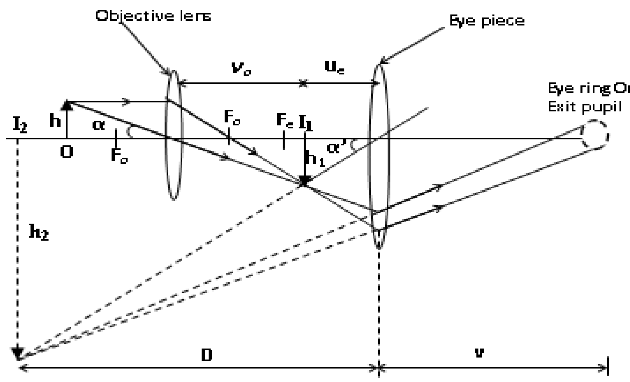
The angular magnification produced in this case is smaller than when the image is formed at the near point.

The only advantage is that the eye is relaxed (un accommodated).

***Eye ring/ Exit pupil***

For a given objective lens, the best position of the observer's eye is that where it collects all rays coming from the objective lens. This position is called the *eye ring or exit pupil*.

Hence eye ring is the position of the image of the objective formed by the eye piece.




Note: when determining the eye ring, the separation is taken as the object distance and focal length of the eye piece is used in calculations. Hence from the above:

$u = v_0 + u_e;\ f = f_e$

Then use;

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

$$\frac{1}{f_e} = \frac{1}{u_e + v_0} + \frac{1}{v}$$

Hence ***v***, which is the eye ring can be obtained.

**Examples:**

4. UNEB 2006 No.3 (b)

An object of size 2.0mm is placed 3cm in front of the objective of a compound microscope. The foal length of the objective is 2.5cm while that of the eye piece is 5.0cm. The microscope forms a virtual image of the object at the near point of the eye. Find the;

(i) Size of the final image.

(ii) Position of the eye ring.

**Solution:**

(i) Size of the final image.

Near point = 25cm. Hence, v for the eye piece = -25cm since the final image is virtual.

$Magnification\ of\ the\ eye\ piece,$

$M_e = \left(\frac{v_e}{f_e} - 1\right)\ or\ \ M_e = \frac{v_e}{u_e}$

$M_e = \left(\frac{-25}{5} - 1\right) = -6$

Action of the Objective:

$\frac{1}{f_0} = \frac{1}{u_0} + \frac{1}{v_0}$

$\frac{1}{2.5} = \frac{1}{3} + \frac{1}{v_0}$

$v_0 = 15\text{cm}$

$Magnification\ of\ the\ objective,$

$$M_0 = \left(\frac{v_0}{f_0} - 1\right) \quad or \quad M_0 = \frac{v_0}{u_0}$$
$$M_e = \left(\frac{15}{2.5} - 1\right) = 5$$

$Effective\ Magnification\ of\ the\ system, M$
$$M = M_0 M_e$$
$$M = (-6)(5) = -30$$
The negative shows that the final image is virtual.

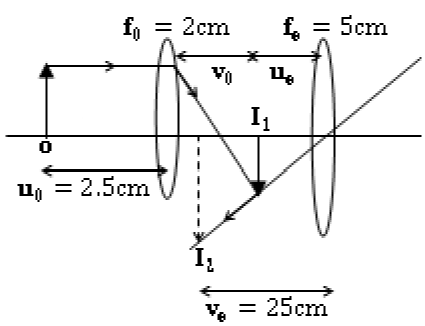
$$M = \frac{h_2}{h_0}$$
Where, $h_0$ is the height of the object, and $h_2$ is the height of the final image.
$$30 = \frac{h_2}{2} \Leftrightarrow h_2 = 2(30) = 60mm$$

(ii) Position of the eye ring.
Since, $M_e = 6$ and $M_e = \frac{v_e}{u_e}$ where $u_e$ is the distance of the intermediate image from the eye piece.
$$6 = \frac{25}{u_e} \Leftrightarrow u_e = 4.17cm$$

Note: The eye ring is the image of the objective in the eye piece.
$u = Lens\ seperation = (15 + 4.17) = 19.17cm$
$v = ?$
$$\frac{1}{f_e} = \frac{1}{u} + \frac{1}{v}$$
$$\frac{1}{5} = \frac{1}{19.17} + \frac{1}{v}$$
$v = 6.76cm$
Thus the position of the eye ring is 6.67 cm from the eye piece.

**Example; UNEB 2000 No. 1(b)**
The objective of a compound microscope has a focal length of 2cm while the eye piece has a focal length of 5cm. An object is placed at a distance of 2.5 cm in front of the objective. The distance of the eye piece from the objective is adjusted so that the final image is 25cm in front of the eye piece. Find the distance between the objective and the eye piece.

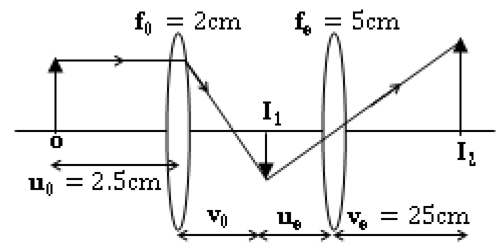
**Solution:**




| Action of the Objective: | Action of the Eye piece: |
|---|---|
| $\frac{1}{f_0} = \frac{1}{u_0} + \frac{1}{v_0}$<br>$\frac{1}{2} = \frac{1}{2.5} + \frac{1}{v_0}$<br>$v_0 = 10cm$ | $\frac{1}{f_e} = \frac{1}{u_e} + \frac{1}{v_e}$<br>$\frac{1}{5} = \frac{1}{u_e} + \frac{1}{-25}$<br>$u_e = 4.167cm$ |

Hence the separation, d is given by;
$d = v_0 + u_e$
$d = 10 + 4.167$
$d = 14.167cm$

<u>Alternatively, if the final image is real, then;</u>




| Action of the Objective: | Action of the Eye piece: |
|---|---|
| $\frac{1}{f_0} = \frac{1}{u_0} + \frac{1}{v_0}$<br>$\frac{1}{2} = \frac{1}{2.5} + \frac{1}{v_0}$<br>$v_0 = 10cm$ | $\frac{1}{f_e} = \frac{1}{u_e} + \frac{1}{v_e}$<br>$\frac{1}{5} = \frac{1}{u_e} + \frac{1}{25}$<br>$u_e = 6.25cm$ |

Hence the separation, d is given by;
$d = v_0 + u_e$
$d = 10 + 6.25$
$d = 16.25cm$

**NOTE:**
- ✓ Always try to sketch the problem in a diagram,
- ✓ Always work out numerical problems from 1st principles not formulae.

**Exercise:**
1. A compound microscope consists of an objective and an eye piece of focal length 2cm and 5cm respectively. The distance between the lenses is 15cm and the final image is formed 25cm from the eye piece. Calculate the;
   (i) Position of the object.[$u_0 = 2.5cm$]
   (ii) Magnifying power of the microscope.[$M = 26.5$]

2. A compound microscope consists of 2 thin converging lenses. The focal length of the objective is 10mm while that of the eye piece lens is 20mm. If an object is placed 11mm from the objective lens, the instrument produces an image at infinity. Calculate the;
   (i) Separation of the lenses. [$d = 130mm$]
   (ii) Magnifying power of the instrument. [$M = 125$]

3. (a) What is meant by magnifying power of an optical instrument.
(b) Derive an expression for the magnifying power, of a compound microscope in normal adjustment.
(c) Why should the objective and eye piece of a compound microscope have short focal length?
(d) Describe with the aid of a ray diagram the structure and action of a compound microscope in normal adjustment.
(e) The objective of a compound microscope has a focal length of 2.0cm while the eye piece has a focal length of 5.0cm. An object is placed at a distance of 2.5cm in front of the objective. The distance of the eye piece from the objective is adjusted so that the final image is 25cm in front of the eye piece. Find the separation of the two lenses. [14.2cm].

4. A compound microscope has an objective of 5cm and an eye piece of focal length 10cm separated by a distance of 16cm. Find the magnification of the image formed at a distance of 25cm from the eye piece. [M=1.158]

5. UNEB 2009 No.1(c)

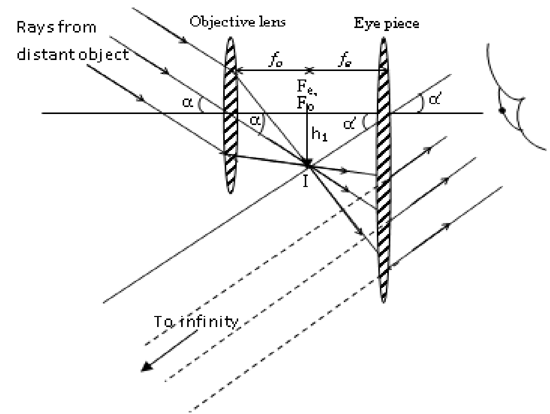
# TELESCOPES

Telescopes are used to view distant objects.
Its structure is composed of the objective of long focal length and an eye piece of short focal length.
The angular magnification of a telescope is the ratio of the angle subtended by the final image at the aided eye to the angle subtended by the object at the un aided eye.

Since the rays incident on the objective are from a distant object, the telescope forms an intermediate image, **I** at the principle focus of the objective lens, $\mathbf{F_0}$.
In normal adjustment, the final image is at infinity. Thus at the principle focus of the eye piece lens, $\mathbf{F_e}$ must coincide with $\mathbf{F_0}$.
The eye of the observer is fully accommodated and it suffers minimum deviation.
However when the final image is at the near point, (Not in normal adjustment), $\mathbf{F_e}$ does not coincide with $\mathbf{F_0}$.

<u>REFRACTING TELESCOPES</u>
1. ASTRONOMICAL TELESCOPE:
This consists of a convex lens of long focal length which acts as its objective and another convex lens of short focal length which acts as its eye piece.

***Astronomical telescope in normal adjustment***




In normal adjustment, the image of the distant object formed by the objective lens lies in the focal plane of both the objective and the eye piece.
Hence separation between the lenses,
$\boldsymbol{d = f_0 + f_e}$

Angular magnification;

$$M = \frac{\alpha^1}{\alpha}$$

For the aided eye, and for small angles in radians,

$$\alpha' \approx \tan\alpha' = \frac{h_1}{f_e}$$

For an unaided eye, and for small angles in radians,

$$\alpha \approx \tan\alpha = \frac{h_1}{f_0}$$

Angular magnification, $M = \frac{\alpha^1}{\alpha}$

$$\mathbf{M} = \frac{\alpha^1}{\alpha} = \frac{h_1}{f_e} \div \frac{h_1}{f_0} = \frac{h_1}{f_e} \times \frac{f_0}{h_1} = \frac{\mathbf{f_0}}{\mathbf{f_e}}$$

$$\mathbf{M} = \frac{\mathbf{f_0}}{\mathbf{f_e}}$$

$$M = \frac{\text{Focal length of objective lens}}{\text{Focal length of the eye piece lens}}$$

**The eye ring and relation to angular magnification**
$u = f_0 + f_e$: $\quad f = f_e$ ; $v = ?$

Then use;

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

$$\frac{1}{f_e} = \frac{1}{f_e + f_0} + \frac{1}{v} \Leftrightarrow \frac{1}{v} = \frac{1}{f_e} - \frac{1}{f_e + f_0}$$

$$\frac{1}{v} = \frac{f_0}{f_e(f_e + f_0)}$$

$$v = \frac{f_e}{f_0}(f_0 + f_e)$$

This gives the position of the eye ring.

From Linear Magnification of the eye ring

$$m = \frac{v}{u} = \frac{\frac{f_e(f_0+f)}{f_0}}{(f_0 + f_e)} = \frac{f_e}{f_0}$$

$$m = \frac{f_e}{f_0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Similarly, linear magnification, m of the eye ring is;

$$m = \frac{Image\ height}{Object\ height}$$

$$m = \frac{Diameter\ of\ the\ eye\ ring}{Diameter\ of\ the\ objective\ lens}$$

$$m = \frac{D_e}{D_0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Equating equations (i) and (ii),

$$m = \frac{f_e}{f_0} = \frac{D_e}{D_0}$$

But Angular magnification, M is given by;

$$\mathbf{M} = \frac{\mathbf{f_0}}{\mathbf{f_e}} = \frac{\mathbf{D_0}}{\mathbf{D_e}}$$

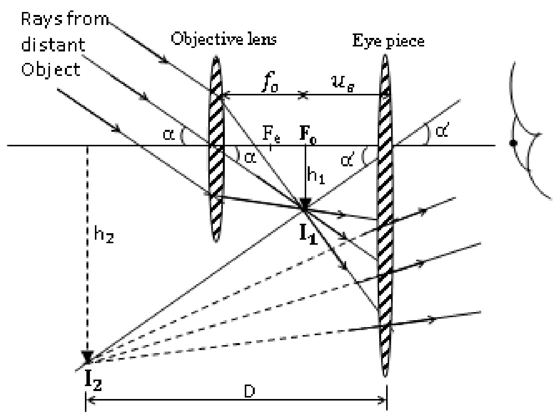
This holds only when the telescope is in normal adjustment.

***Astronomical telescope with image formed at near point (Not in normal adjustment)***

In this case, the intermediate image, I is still in the focal plane of the objective lens but nearer the eye piece than its focal point, $\mathbf{F_e}$. The intermediate image should be formed in front of the focal point of the eye piece. Therefore, the eye piece acts as a magnifying glass

Hence separation between the lenses,

$$\boldsymbol{d = f_0 + u_e}$$




In this case, the eye ring is located between the two lenses hence it is inaccessible to the eye.

Thus the eye cannot collect all the light passing through the objective hence the image does not appear brighter.

Angular magnification;

$$M = \frac{\alpha'}{\alpha}$$

For the aided eye, and for small angles in radians,

$$\alpha' \approx \tan\alpha' = \frac{h_2}{D}$$

For an unaided eye, and for small angles in radians,

$$\alpha' \approx \tan\alpha = \frac{h_1}{f_0}$$

Angular magnification, $M = \frac{\alpha^1}{\alpha}$

$$\mathbf{M} = \frac{\alpha^1}{\alpha} = \frac{h_2}{D} \div \frac{h_1}{f_0} = \frac{h_2}{D} \times \frac{f_0}{h_1} = \frac{h_2}{h_1} \times \frac{f_0}{D}$$

But $\frac{h_2}{h_1}$ is linear magnification of the eye piece;

$m_e = \frac{h_2}{h_1} = \left(\frac{D}{f_e} - 1\right)$. Hence;

$$M = \frac{f_0}{D}\left(\frac{D}{f_e} - 1\right)$$

$$\boldsymbol{M = \frac{f_0}{f_e}\left(1 - \frac{f_e}{D}\right)}$$

Considering D as negative, the angular magnification becomes;

$$M = \frac{f_0}{f_e}\left(1 + \frac{f_e}{D}\right)$$

**Note:** The magnifying power of an astronomical NOT in normal use is greater than that when it is in normal use.

***Advantages of Astronomical Telescopes:***

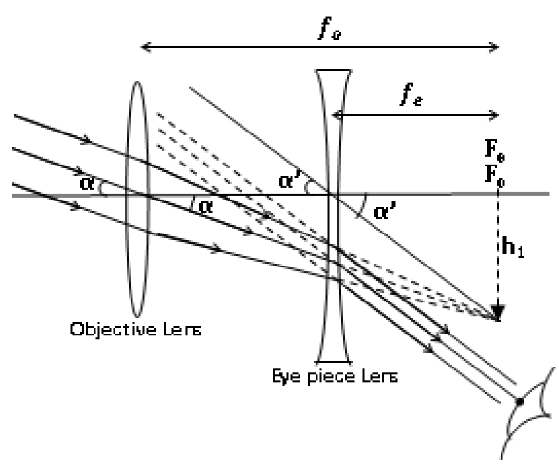
(i) It forms an erect image of a distant object
(ii) It is less bulky as it is more compact
(iii) It has a wider field of view.

## 2. GALILEAN TELESCOPE

A Galilean telescope consists of a convex objective lens and a concave eye piece lens. The objective has a longer focal length than the eye piece.

Hence separation between the lenses,

$$\boldsymbol{d = f_0 - f_e}$$




The intermediate image is the image of the distant object in the objective. It acts as a virtual object for the eye piece lens which forms the final erect image at infinity.

Angular magnification;

$$M = \frac{\alpha^1}{\alpha}$$

For the aided eye, and for small angles in radians,

$$\alpha' \approx \tan\alpha' = \frac{h_1}{f_e}$$

For an unaided eye, and for small angles in radians,

$$\alpha \approx \tan\alpha = \frac{h_1}{f_0}$$

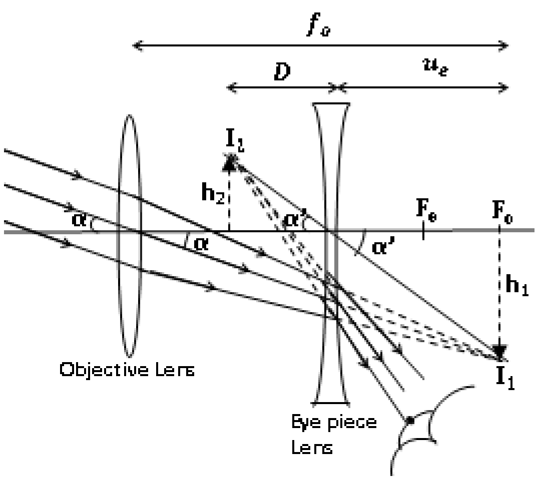
Angular magnification, $M = \frac{\alpha^1}{\alpha}$

$$\mathbf{M} = \frac{\alpha^1}{\alpha} = \frac{h_1}{f_e} \div \frac{h_1}{f_0} = \frac{h_1}{f_e} \times \frac{f_0}{h_1} = \frac{\mathbf{f_0}}{\mathbf{f_e}}$$

$$\mathbf{M} = \frac{\mathbf{f_0}}{\mathbf{f_e}}$$

**Exercise:**
Draw a Galilean telescope with the final image at the near point and show that the angular magnification is the same as that of an astronomical telescope.
Hence separation between the lenses,

$$\boldsymbol{d = f_0 - u_e}$$




Angular magnification;

$$M = \frac{\alpha^1}{\alpha}$$

For the aided eye, and for small angles in radians,

$$\alpha' \approx \tan\alpha' = \frac{h_1}{u_e}$$

For an unaided eye, and for small angles in radians,

$$\alpha' \approx \tan\alpha = \frac{h_1}{f_0}$$

Angular magnification, $M = \frac{\alpha^1}{\alpha}$

$$\mathbf{M} = \frac{\alpha^1}{\alpha} = \frac{h_1}{u_e} \div \frac{h_1}{f_0} = \frac{h_1}{u_e} \times \frac{f_0}{h_1} = \frac{f_0}{u_e}$$

$$\mathbf{M} = \frac{\mathbf{f_0}}{\mathbf{u_e}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\boldsymbol{i})$$

Using the lens formula;

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

$$\frac{1}{f_e} = \frac{1}{u_e} + \frac{1}{D}$$

$$\frac{1}{u_e} = \frac{1}{f_e} - \frac{1}{D} \Leftrightarrow \frac{1}{u_e} = \frac{D - f_e}{Df_e}$$

$$\frac{1}{u_e} = \frac{D - f_e}{Df_e} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Putting equation (ii) into equation (i):

$$\boldsymbol{M = f_0}\left(\frac{D - f_e}{Df_e}\right)$$

$$M = \frac{f_0}{f_e}\left(1 - \frac{f_e}{D}\right)$$

Using real is positive and virtual is negative, $f_e$ *and* $D$ are both negative.

***Advantages of Galilean Telescope:***

(i) It is shorter than astronomical telescope when in normal adjustment, hence it used for *opera glasses*

(ii) The final image is upright or erect.

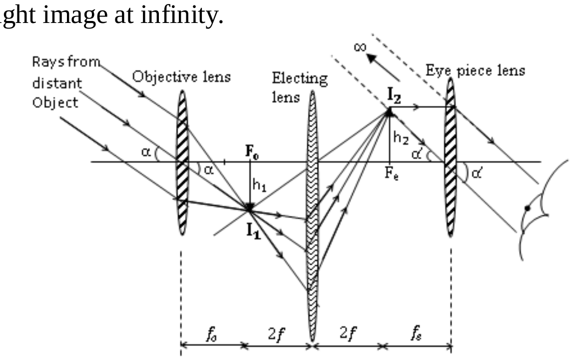
***Disadvantages of Galilean Telescope:***

(i) It has a virtual eye ring. Thus the observer must put the eye as close as possible to the eye piece. However, this reduces the field of view.

(ii) It has a limited field of view.

## 3. TERRESTRIAL TELESCOPE

It is a refracting telescope with an intermediate erecting lens of focal length, f placed between the objective and the eye piece.
The electing lens is placed at a distance of 2f in front of the inverted image $I_1$ formed by the objective.
The image $I_2$ formed by the electing lens is upright, real, the same size as $I_1$ and at a distance of 2f from the electing lens and at the principle focus of the eye piece. It thus forms a final upright image at infinity.




In normal adjustment, the angular magnification, M is given by;

$$\mathbf{M} = \frac{\mathbf{f_0}}{\mathbf{f_e}}$$

***Advantage of terrestrial telescope***

(i) It produces upright images with respect to the object.

***Disadvantages of terrestrial telescope***

(i) Erecting lens reduce the intensity of light emerging from the eye piece, as light is reflected at the lens surfaces. Hence the final image is less bright.

(ii) Separation$(4f + f_0 + f_e)$of lenses is longer than any other telescope.

**Differences between compound microscopes and Astronomical Telescopes**.

| Compound microscope | Astronomical telescope |
|---|---|
| -Used to see tinny nearby objects | -Used to see distant objects |
| -In normal adjustments, the final image is at near point. | -In normal adjustments, the final image is at infinity. |
| -The objective and eye piece lenses have short focal length. | The objective and eye piece lenses have long focal length. |
| -The focal length of the objective is less than that of the eye piece. | -The focal length of the objective is greater than that of the eye piece. |
| -The visual angle is given by $\propto = \frac{h}{D}$ | -The visual angle is given by $\propto = \frac{h}{f_0}$ |

**Example; 1**

The objective of an astronomical telescope in normal adjustment has a diameter of 15cm and focal length 400cm. The eye piece has a focal length 2.5cm.

(a) Find the;

(i) Position of the eye ring

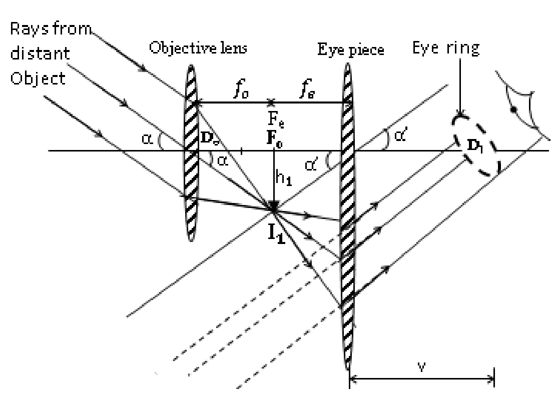
(ii) Diameter of the eye ring

(iii) Magnifying power of the eye ring

(b) Give one advantage of placing the eye at the eye ring.

**Solution:**

**(a)**




(i) $f_0 = 400\text{cm}, f_e = 2.5\text{cm}, D_0 = 15\text{cm}.$

The position of the eye ring:

$u = f_0 + f_e$

$u = 400 + 2.5$

$u = 402.5\text{cm}$

Using the lens formula;

$$\frac{1}{f_e} = \frac{1}{u} + \frac{1}{v}$$

$$\frac{1}{2.5} = \frac{1}{402.5} + \frac{1}{v}$$

$v = 2.52$ cm

(ii) The diameter of the eye ring;

$$M = \frac{f_0}{f_e} = \frac{D_0}{D_I}$$

$$\frac{400}{2.5} = \frac{15}{D_I}$$

$$D_I = \frac{15 \times 2.5}{400} = 0.09375\text{m}$$

$$D_I = 0.09375\text{cm}$$

(iii) Magnifying power:

$$M = \frac{f_0}{f_e} = \frac{400}{2.5} = 160$$

(iv) When the eye is placed at the eye ring, it receives all the light falling onto the objective.

**Example; 2**

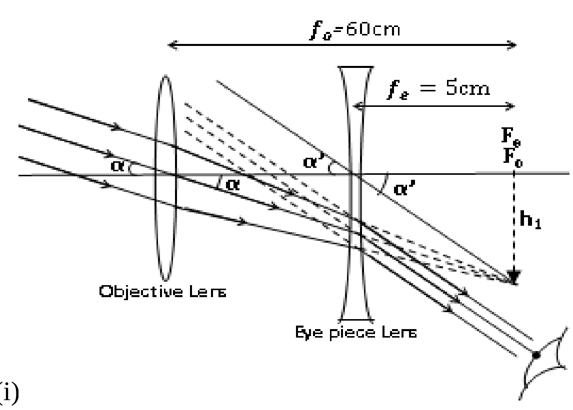
A convex lens of focal length 60cm is arranged coaxially with a diverging lens of focal length 5cm to view a distant star.

(i) If the final image is at infinity, draw a ray diagram to show the formation of the image of the star.

(ii) Calculate the magnifying power obtained if the image of the star is formed at a distance of 25cm in front of the eye piece.

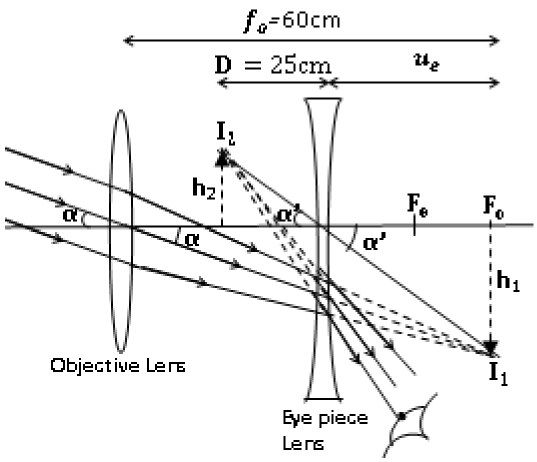
(iii) List one advantage and one dis advantage of this type of arrangement over an astronomical telescope.

**Solution:**




(i)

(ii)




$$\frac{1}{f_e} = \frac{1}{u_e} + \frac{1}{D}$$

$$\frac{1}{-5} = \frac{1}{u_e} + \frac{1}{-25}$$

$$\frac{1}{u_e} = \frac{-4}{25}$$

Angular magnification;

$$M = \frac{á^1}{á}$$

$$M = \frac{h_1}{u_e} \div \frac{h_1}{f_0}$$

$$M = \frac{f_0}{u_e} = 60 \times \frac{-4}{25}$$

$$M = 9.6$$

**Example; 3**

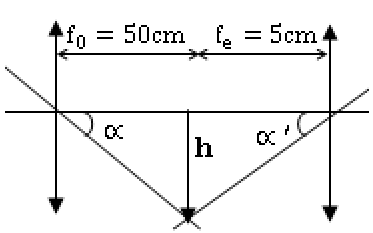
An astronomical telescope in normal adjustment has an objective of focal length 50cm and an eye piece lens of focal length 5cm.

(i) What is its angular magnification?

(ii) If it is assumed that the eye is placed very close to the eye piece, and that the pupil of the eye has a diameter 3mm, what will be the diameter of the objective if all light passing through the objective is to emerge as a beam which fills the pupil of the eye?

**Solution:**

**(i)**




$$\propto \approx \tan \propto = \frac{h}{50}; \qquad \propto' \approx \tan \propto' = \frac{h}{5};$$

Angular magnification;

$$M = \frac{á^1}{á}$$

$$M = \frac{h}{5} \div \frac{h}{50} = \frac{h}{5} \times \frac{50}{h} = 10$$

$$M = 10$$

(ii) The diameter of the objective;

$$M = \frac{f_0}{f_e} = \frac{u}{v} = \frac{D_0}{D_I}$$

$$M = \frac{D_0}{D_I}$$

$$10 = \frac{D_0}{3}$$

$$D_0 = 30\text{mm}$$

<u>Thus the diameter of the objective is 30mm.</u>

**Example; 4**

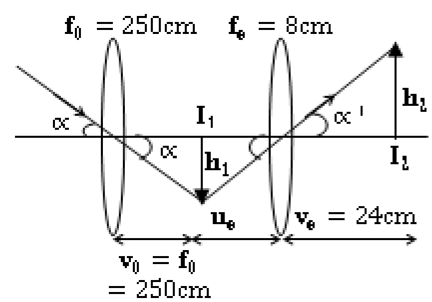
A simple astronomical telescope has an objective of focal length 250cm and an eye piece of 8cm. The eye piece is adjusted so that **a real image** of the sun is formed on a screen placed 24cm from the eye piece.

(a) Calculate the distance between the objective and the eye piece.

(b) If the suns image on the screen has a diameter of 10cm, calculate the angle subtended by the sun on the objective.

**Solution:**

**(a)**




Action of the eye piece

$$\frac{1}{f_e} = \frac{1}{u_e} + \frac{1}{v_e}$$

$$\frac{1}{8} = \frac{1}{u_e} + \frac{1}{24}$$

$$\frac{1}{u_e} = \frac{2}{24} \Leftrightarrow u_e = 12\text{cm}$$

Distance between the lenses;

$$d = f_0 + u_e$$

$$d = 250 + 12$$

$d = 262$ cm

(b)

$\propto \approx \tan \propto = \frac{h_1}{250}; \propto' \approx \tan \propto' = \frac{h_2}{8};$

$$m = \frac{h_2}{h_1} = \frac{v_e}{u_e} \Leftrightarrow \frac{10}{h_1} = \frac{24}{12} \Leftrightarrow h_1 = 5\text{cm}$$

Thus from;

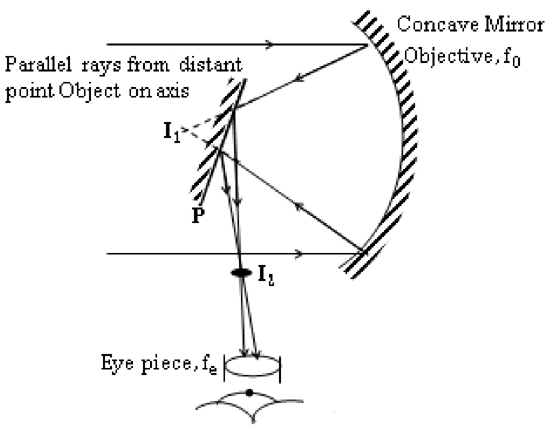
$\propto \approx \tan \propto = \frac{h_1}{250} = \frac{5}{250} = 0.02$ rad

Therefore, the angle subtended by the sun at the objective is 0.02 rad.

## REFLECTING TELESCOPES

***(i) Newton's reflecting telescope***

It consists of a concave mirror of long focal length as the objective instead of a convex lens, a plane mirror and a convex eye piece.

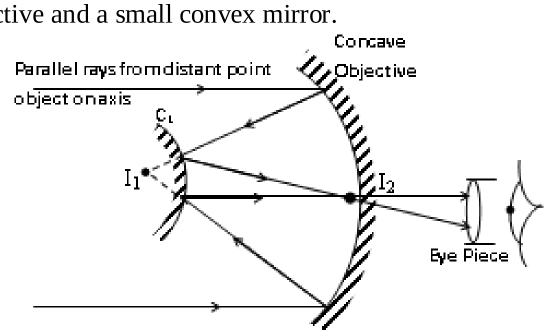



The objective mirror reflects rays from the object onto the plane mirror, P which reflects the rays onto the eye piece lens through which the image is observed. In normal adjustment, the final image is inverted and angular magnification,

$$M = \frac{f_o}{f_e}$$

**(ii) Cassegrain reflecting telescope**

It consists of a concave mirror of long focal length as the objective and a small convex mirror.




Light rays from a distant object are reflected from a concave objective and also from a small convex mirror $C_1$.

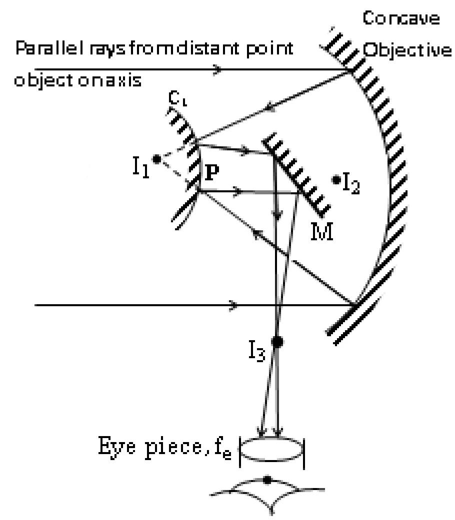
The objective forms an intermediate image $I_1$ which acts as a virtual object for $C_1$.

The reflected rays pass through a hole at the pole of concave objective, forming an intermediate image $I_2$ which can be magnified by the eye piece.

$$M = \frac{f_o}{f_e}$$

**(iii) Cloude' reflecting telescope**

This is a combination of Newtonian and Cassegrain reflecting telescopes.




Parallel rays of light from a distant object are reflected from the concave objective of long focal length, at the convex reflector, $C_1$ and finally at the plane mirror, M to form a final image $I_3$ which is magnified by the eye piece.

In normal adjustment, the final image is formed at infinity.

$$m_{convex} = \frac{PI_2}{PI_1}$$

Where, P is the pole of the convex mirror.

The effective magnification of the telescope in normal adjustment is given by;

$$\boldsymbol{M_{power}} = \left(\frac{\boldsymbol{PI_2}}{\boldsymbol{PI_1}}\right)\left(\frac{\boldsymbol{f_0}}{\boldsymbol{f_e}}\right)$$

***Advantages of reflecting telescope over refracting telescope***

(i) There is no chromatic aberration since no refraction occurs at the objective which is a mirror.

(ii) Spherical aberration can be reduced easily by using a parabolic mirror as the objective.

(iii) It is much cheaper and easier to make a mirror than a lens since one surface requires to be grounded.

(iv) Images are brighter because it is easy to make mirrors of large aperture which collects a lot of light compared to making a lens with large aperture

(v) Resolving power is also greater ( i.e. seeing different images as separate) since their objective can be made to have a large diameter

(vi) They are portable.

***Disadvantage of reflecting telescope over refracting telescope.***

(i) They tarnish easily and absorb light resulting in dull images.

**Example:1**

A distant object subtending an angle of $3 \times 10^{-5}$ radians is viewed with a reflecting telescope whose objective is a concave mirror of focal length 10m. The reflected light is intersected by a convex mirror placed 9.5m from the pole of the objective and a real image is formed at a hole at the pole of the objective.

The image viewed with a convex lens of focal length 5cm used as a magnifying glass producing the final image at infinity.

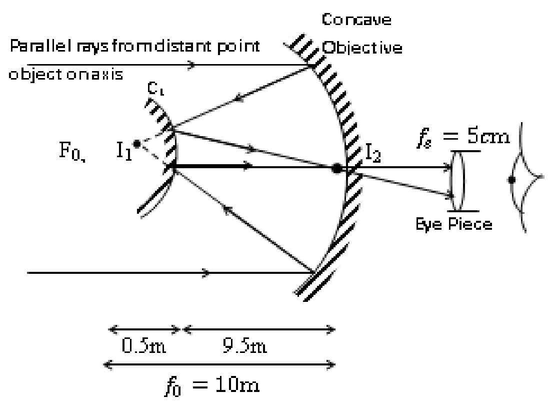
(i) Draw a ray diagram for the arrangement of the telescope.

(ii) Calculate the diameter of the real image formed at pole of the concave mirror.

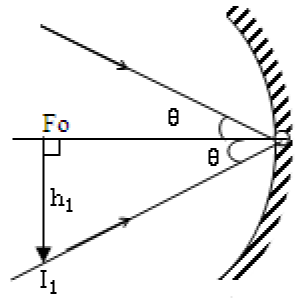
(iii) Determine the angular magnification of the telescope.

**Solution:**

**(i)**




**(ii)** The diameter of the image $I_1$ of the distant object formed at the focal point of the objective is given by;




Where, $\text{è} = 3 \times 10^{-5}$ radians

$\tan\theta = \frac{h_1}{f_0}$

But for small angles in radians, $\tan\theta \approx \theta$

$\Leftrightarrow \text{è} = \frac{h_1}{f_0}$

$\Leftrightarrow h_1 = \text{è}f_0$

$\Leftrightarrow h_1 = (3 \times 10^{-5}) \times 10$

$\Leftrightarrow h_1 = 3 \times 10^{-4}\text{m}$

$\text{Magnification}, = \frac{h_2}{h_1} = \frac{v}{u}$

$\frac{h_2}{3 \times 10^{-4}} = \frac{9.5}{0.5}$

$h_2 = 0.0057\text{m}$

(iv)

The angular magnification of the arrangement;

$M = m_1 m_2$

$M = \left(\frac{f_0}{f_e}\right)\left(\frac{h_2}{h_1}\right)$

$M = \left(\frac{f_0}{f_e}\right)\left(\frac{v}{u}\right)$

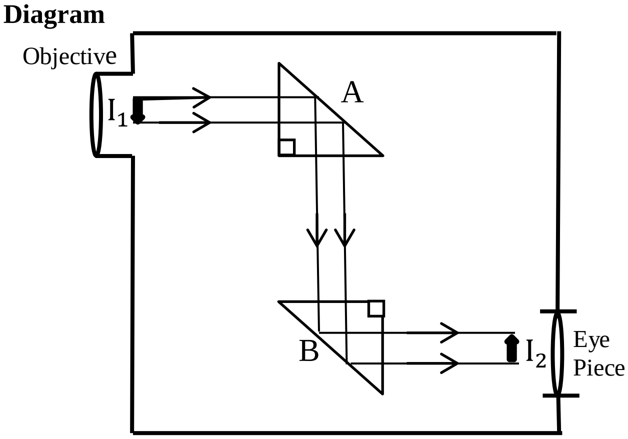
$M = \left(\frac{10}{5 \times 10^{-2}}\right)\left(\frac{9.5}{0.5}\right)$

$M = 3800$

## THE PRISM BINOCULARS

It consists of a pair of a reflecting astronomical telescope with two reflecting prisms between each objective and eye piece

**Diagram**




**Action**

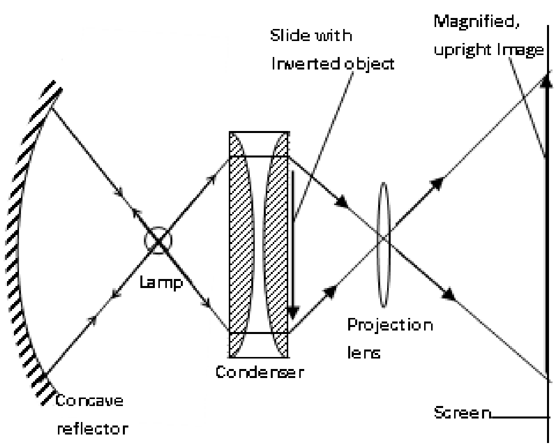
The objective lens forms a real inverted image$I_1$, of the distant object at its principal focus.
Image $I_1$ is in front of the right angled prism A.
By total internal reflection, the prism A causes a lateral inversion direction through $180^0$.
The right angled prism B, then inverts the image such that an erect image $I_2$ of the same size as $I_1$ is formed in front of the eye piece.
$I_2$ is the same way up as the distant object.
The eye piece then forms a virtual magnified image of $I_2$
The arrangement of the prisms makes the binoculars shorter than an astronomical telescope of the same objective and eye piece. It is about $\frac{1}{3}$ of the optical distance.

**The magnification**
The magnifying power is the same as that of the astronomical telescope 3 times as long and in the same adjustment. This is because the optical path of the light is about 3 times the distance between the objective and the eye piece.

***Advantages of Prism binoculars over telescope***
-Binoculars are shorter than telescopes
-They produce upright images
-They have a wide field of view.

## PROJECTION LANTERN

A projector is designed to throw on a screen a magnified image of a film or transparency. It consists of an illumination system and a projection lens.




***Functions of the parts of the projector***
(i) The concave reflector concentrates light from the source on to the condenser.
(ii) The condenser uniformly illuminates the object with light from the source. Or it concentrates light toward the object.
(iii) The slide bears the object to be projected to the screen
(iv) Projection lens focuses and magnifies an upright image on the screen
(vii) Screen is where the image is viewed.

**Example 1**
A projection lantern has an objective lens of focal length 30cm. The area of the picture on its slide is 10cm$^2$. The projector is required to form pictures on the screen 2m$^2$ and 4m$^2$ at different instants. Determine the distance through which the objective lens should be moved to obtain the required size of the image.
**Solution**
Area of object $= A_o = 10m^2$
Area of image $= A_I = 2m^2 = 200000cm^2$
Area magnification$= \frac{A_I}{A_o} = \frac{20000}{10} = 2000$
Linear magnification$= \sqrt{area.magnification}$
$= \sqrt{2000}$
Now using; $\frac{1}{m} = \frac{u}{f} - 1$
$$\frac{1}{\sqrt{2000}} = \frac{u}{30} = 1$$
$u = 30.671cm$

When area of image $= 4m^2 = 40000cm^2$
Therefore linear magnification $= \sqrt{4000}$
Again using; $\frac{1}{m} = \frac{u}{f} - 1$
$$\frac{1}{\sqrt{4000}} = \frac{u}{30} = 1$$
$u = 30.474cm$
Distance of adjustment of the lens
$= 30.671 - 30.474$
$= 0.1967cm$

**Example 2**
A slide projector has a lens of focal length 30cm. determine the distance through which the lens must be moved so that the screen distance varies from 12m to 18m.

**Solution:**
$f = 30cm$
$when\ V = 12m = 1200cm, we\ obtain\ the\ value\ of\ u$

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$
$$\frac{1}{u} + \frac{1}{1200} = \frac{1}{30} \Rightarrow 30.769cm$$
$when\ V = 18cm = 1800cm$
$$\frac{1}{u} + \frac{1}{1800} = \frac{1}{30}$$
$u = 30.508cm$

Therefore distance of movement of the lens
$= 30.769 - 30.508$
$= 0.261cm$

**Question:**
A projection lantern has an objective lens of focal length 20cm. It is required to project clear images on the screen for distances between the slide and the screen ranging from8m to 14m. Calculate the;

(i) Displacement of the lens required to achieve proper focusing between the two extremes.

(ii) Ratio of the magnification at the two extremes.

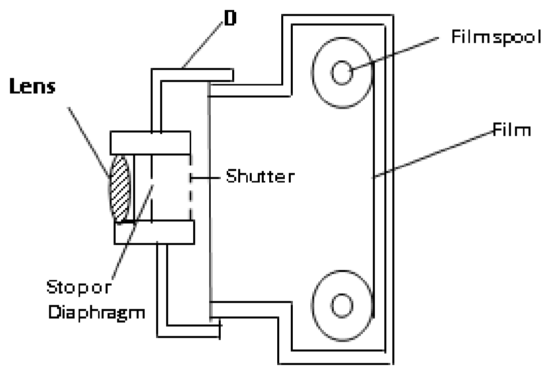
## THE LENS CAMERA

**The Lens Camera**

It consists of a lens system, a focusing device to adjust the distance of the lens for proper focusing, an exposure system and a light sensitive film.

The lens in the camera is achromatic doublet; hence it is free from chromatic aberration.

The stop or aperture is provided so that the light is incident centrally on the lens, thus diminishing distortion.




The light from the object passes through the lens system which consists of an achromatic doublet and separate half lenses in order to minimize chromatic aberration and spherical aberration.

The focusing device adjusts or varies the distance of the stop to ensure that the light passes through the central region of the lens system and this minimizes distortions in the images formed in the film.

When the photograph is taken, the shutter opens so that the film is exposed to the light from the object. The shutter also ensures that there are the correct times of exposure for a given aperture size.

An inverted image of the object is formed on the light sensitive film.

The aperture should be small so that;

1. Spherical aberration is reduced since the light would then be passing only through the center of the lens system.

2. The depth of field is increased

**Similarities and differences between the eye and lens camera.**

**1. Similarities**

The camera consists of the (a) light proof box painted black inside the eye it is fitted with a black pigment in to it to prevent stray reflection of light

Both have converging lens that focus light from the external objects;

Both have light sensitive parts, the camera has a film while the eye has a retina.

Both have a system that controls the amount of light entering them.

In the eye, iris is responsible and diaphragm does the same function in the camera.

**2. Differences**

-The eye lens is a biological organ while that of a camera is made out of glass.

-The distance between the eye lens and the retina is fixed while that between the camera lens and the film can be varied.

-The eye focuses image by changing the shape of the lens, in a camera the image is focused by changing the distance between the lens and the film.

**Example:**

A converging lens is used to form an image of an object 1.2m away on a film 0.05m from the lens.

(i) Find the focal length of the lens.

(ii) If the camera is to be used to photograph a distant object, how far from the film would a clear image be formed?

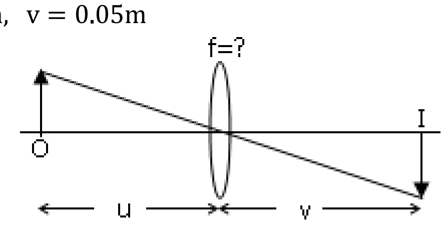
(iii) State the type of lens that would be placed close to the first lens in order to enable the image in (ii) above to be formed on the film.

(iv) Calculate the focal length of the lens you have stated in (iii) above.

**Solution:**

**(i)**

$u = 1.2\text{m},\ \ v = 0.05\text{m}$



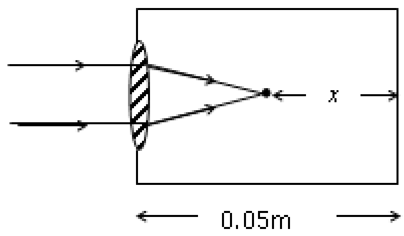

Using the lens formula;

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

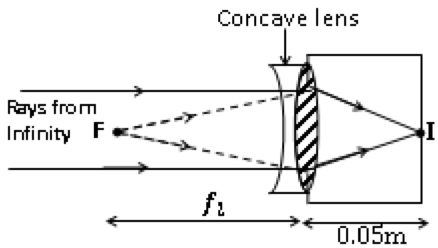
$$\frac{1}{f} = \frac{1}{1.2} + \frac{1}{0.05}$$

$$f = 0.048\text{m}$$

(ii)




$Distance, x = v - f$
$x = 0.05 - 0.048$
$x = 0.002$m
(iii)
A diverging lens (or concave lens)




Using the lens formula;
$$\frac{1}{\mathrm{F}} = \frac{1}{f_1} + \frac{1}{f_2}$$

$$\frac{1}{0.05} = \frac{1}{0.048} + \frac{1}{f_2}$$
$f_2 = -1.2$m
Thus the diverging lens is of focal length 1.2m

**Exercise:**
1. UNEB 2000, 1993, 1992
2. UNEB 2001 No.1 (d) [d = 107.14cm]
3. An astronomical telescope has an objective of focal length 100cm and an eye piece of focal length 5cm. Calculate the magnifying power when the final image is at;
(i) Infinity. [M=20]
(ii) Near point. [M=24]

# CLASSIFICATION AND GENERAL PROPERTIES OF WAVES

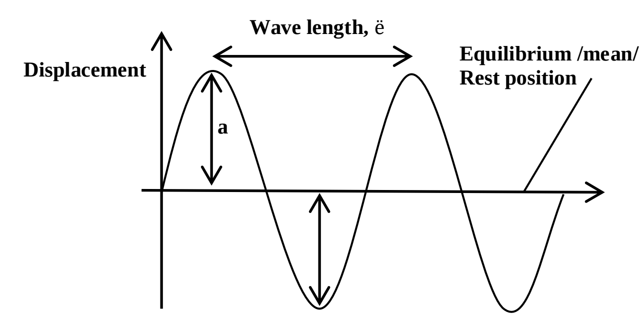
A wave is a periodic disturbance which travels with finite velocity through a medium and remains unchanged in type as it travels. Or it is a disturbance which travels through a medium, transfers energy from one location to another without transferring matter.
Waves may be classified as mechanical or electromagnetic waves.

Mechanical waves: These are waves that require a material medium for their propagation. These include water waves, sound waves and waves on stretched strings.

Electromagnetic waves: These are waves that don't require a material medium for their propagation. They include radio, infra-red, light, Ultraviolet, X-rays, Gamma rays. Electromagnetic waves travel in a vacuum.
If the disturbance of the source of waves is simple harmonic, the displacement in a given time varies with distance from the source as shown below.

Generation and Propagation of mechanical waves.
Waves are generated when particles of a transmitting medium at any point are disturbed and start vibrating.
As they vibrate, they cause the neighboring particles to vibrate in turn, hence causing the vibrations to continue from the source to other regions in the transmitting medium.
The disturbance thus spreads the source outwards and it constitutes the wave.

Graphical representation of a wave.




**Terms used in waves**,
Amplitude: This is the greatest displacement of any wave particle from its equilibrium position.

Wave length (λ): Is the distance between two successive particles in a wave profile that are in phase. It is the distance between two successive crests or troughs.

Wave front. Is any line or section taken through an advancing wave in which all the particles are in the same phase. OR: It is the imaginary line joining the set of particles that are in phase.

Particles are in phase if they are in the same point in their path at the same time and are moving in the same direction.
The direction of travel of the wave is always at right angles to the wave front.

Cycle or Oscillation: is a complete to and fro motion of a wave. It is equivalent to moving from O to B.

Period (T): The time taken for any particle to undergo a complete oscillation. $T = \frac{t}{n}$.
Frequency (f): The number of oscillations per second. $f = \frac{n}{t}$.

Velocity (v): The distance covered by a wave particle per second in a given direction.

Phase: Is a fraction of a cycle which has elapsed after a particle passing a fixed point.

**Relationship between f and T**
If a wave completes n cycles in time t, then frequency, f is given by:
Frequency, $f = \frac{n}{t}$……………………………..(i)
Period, $T = \frac{t}{n}$............................................(ii)
Eqn (i) x eqn (ii) gives;
$$fT = \left(\frac{n}{t}\right) \times \left(\frac{t}{n}\right) = 1 \Leftrightarrow f = \frac{1}{T}$$

**Relationship between v, λ and f**
If a wave of wavelength λ completes n cycles in time t, then the frequency, f is given by;
Each cycle is a wavelength, $\boldsymbol{\lambda}$**:**
Total distance covered in n-cycles = nλ
$$Speed, v = \frac{Distance}{Time} = \frac{n\lambda}{t} = \left(\frac{n}{t}\right)\lambda, \qquad But \frac{n}{t} = f$$
$$\Leftrightarrow \qquad \boldsymbol{v = f\lambda}$$

**Alternatively,**

If a wave covers a distance, λ, the wavelength, then the time taken is T, the period. Hence speed,

$Speed, v = \frac{\lambda}{T} = \left(\frac{1}{T}\right)\lambda$ ,  $But, \frac{1}{T} = f$

$\Leftrightarrow \quad \boldsymbol{v = f\lambda}$

## TYPES OF WAVES

**(i)Transverse Waves;** It is one in which the particles of the medium propagate by vibrations perpendicular to the direction of travel of the wave.

Examples include: Water waves, electromagnetic waves and waves of a stretched string.

Speed of a transverse waves along a stretched string

The speed v of a transverse waves on a stretched string is independent of the amplitude and frequency of the wave. It depends on the tension, T in the string and the mass per unit length μ. The tension will determine the restoring force on a displaced piece of string and mass per unit length will affect its consequent acceleration.

Using dimension analysis;

$v = kT^x\mu^y$, Where k is a dimensionless constant.

$[v] = [T^x\mu^y]$

$LT^{-1} = (MLT^{-2})^x(ML^{-1})^y$

$LT^{-1} = M^{(x+y)}.L^{(x-y)}.T^{-2x}$

Equating corresponding power, you get;

$$x = \frac{1}{2}, \quad and\ y = -\frac{1}{2}$$

$\Leftrightarrow \mathbf{v = kT^{\frac{1}{2}}\mu^{\frac{-1}{2}}}$

Further analysis gives k = 1.

Hence **;**$\mathbf{v = \sqrt{\frac{T}{\mu}}}$

(ii) **Longitudinal waves**

It is a wave in which the particles of the medium vibrate parallel to the direction of travel of the wave.

The vibrations of the individual particles occur in the same direction as the direction of the travel of the wave.

Examples include sound waves and waves in a stretched spring, waves in pipes, earth quake, e.t.c.

For solids, the speed of longitudinal waves is given by;

$v = \left(\frac{Y}{\rho}\right)^{\frac{1}{2}}$ where Y is young's modulus and ρ is the mass density of a solid.

For fluids, the speed of a longitudinal wave is given by; $v = \left(\frac{B}{\rho}\right)^{\frac{1}{2}}$. ; where $B = -\frac{V\Delta P}{\Delta V}$ is the adiabatic bulk modulus and ρ is the density of fluid.

For gases in particular; $v = \left(\frac{\gamma P}{\rho}\right)^{\frac{1}{2}}$; where P is pressure, $\rho$ is density and $\gamma = \frac{C_P}{C_V}$is the ratio of molar heat capacities.

**Note:**

Longitudinal waves are said to under go propagation by an adiabatic process.

- This is because they propagate by compression and rarefaction.
- During compression, the temperature of the medium rises unless heat is withdrawn from the medium.
- During rarefaction, the temperature of the medium decreases unless heat is supplied into the medium.

The compressions and rarefactions occur so fast that heat does not enter or leave the gas (medium). Hence the process is adiabatic.

Differences between Transverse waves and longitudinal waves.

| **Transverse waves** | **Longitudinal waves** |
|---|---|
| -Particles vibrate perpendicularly to the direction of propagation of the wave profile | - Particles vibrate parallel to the direction of propagation of the wave profile |
| -Consists of crests and troughs | -Consists of compression and rarefaction regions |
| -Under go polarisation | -Cannot be polarised |
| -No variation in particle density occurs along the wave profile. | -Particle density varies along the wave profile. |

**Progressive waves or traveling waves**

A progressive wave consists of a disturbance moving from a source to the surrounding places as a result of which energy is transferred from one point to another.

Both transverse and longitudinal waves are progressive waves. The profile of a progressive wave moves along the speed of the wave. It repeats itself at equal distances. The repeated distance is called the wavelength.

If one point in the medium in which the profile propagates is taken, the profile is seen to repeat itself at equal intervals of time called the period. Vibrations of particles in progressive waves are of the same amplitude and frequency but the phase of the vibration changes for different points along the wave.

Phase difference

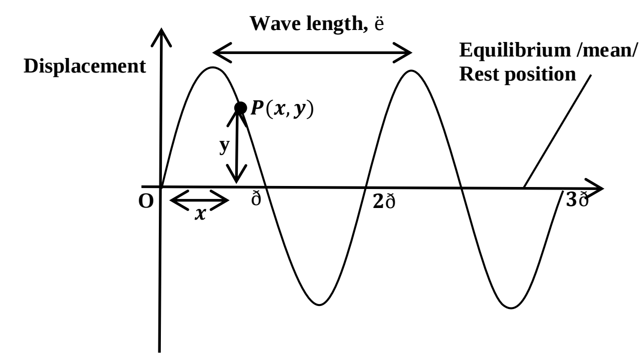
When the crests of two waves of equal wavelength are together, the waves are said to be in phase (i.e. they have a phase difference of zero.)

If a crest and a trough are together, the waves are completely out of phase (i.e. they have a phase difference of $\pi$ radians).

Phase angle: Is the angular displacement between two wave oscillations (or two particles at different points in a wave profile).

Phase difference: Is the difference in the phase angles between any two points along the wave motion.

Suppose the wave moves from left to right and the particles at the origin O vibrate with simple harmonic motion.




The vibrations of the particle at P a distance x from the origin will be out of phase with vibration of the particle at O.
At a distance λ from O corresponds to a phase difference of 2π. i.e.$\lambda \rightarrow 2\pi$

$x \rightarrow \Phi$

$2\pi\, x = \Phi\lambda$

Therefore the phase angle of Φ at P is$\boldsymbol{\Phi = \frac{2\pi x}{\lambda}}$.
The displacement of any particle a distance x from O is given by;$\boldsymbol{y = a\,sin(\omega t - \phi)}$
Where $\phi = \frac{2\pi x}{\lambda}$ is the phase angle, the quantity $\mathbf{k = \frac{2\pi}{\lambda}}$ is called the wave vector or wave number,$\boldsymbol{a}$is the amplitude and $\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi v}{\lambda}$
Hence: $y = a\,sin\left(\omega t - \frac{2\pi x}{\lambda}\right) = a\,sin\left(2\pi f t - \frac{2\pi x}{\lambda}\right)$

$\boldsymbol{y = a\,sin(\omega t - kx)}$

Which is the general equation for a progressive wave moving in the positive x- direction.
.
The negative sign in the equation indicates that vibrations at a point like P, to the right of O will lag behind those at O, for a wave traveling from left to right.
A wave traveling from the right to the left arrives at O before O. hence the vibrations at P would lead that at O.
Hence $\boldsymbol{y = a\,sin(\omega t + kx)}$

**Examples:**

1. The displacement of a wave travelling in x-direction at a time, t is given by;$y\ = A\,sin\,2\pi\left(\frac{t}{0.1} - \frac{x}{2.0}\right)$ m. Find the;
(i) velocity of the wave ($\mathrm{v} = 20\mathrm{m}s^{-1}$)
(ii) Period of the wave (T = 0.1s)
(iii) Progressive wave which will give a stationary wave with the one above. ( $\mathrm{y\ = A\sin 2\pi}\left(\frac{\mathrm{t}}{0.1} + \frac{\mathrm{x}}{2.0}\right)$)
2. Two waves of frequency 256Hz and 280 Hz respectively travel with a speed of 340ms$^{-1}$ through a medium. Find the phase difference at a point 2.0m from where they were initially in phase.($\phi = 0.89$rads)
3. The displacement of a wave travelling in x-direction at a time, t is given by;$y = 2\,sin\,2\pi(0.25x - 100t)$ m. Where x and y are in cm and t in seconds.
Find the; (i) wave length of the wave($\lambda = 4.0$cm)
(ii) Velocity of the wave($\mathrm{v} = 400\mathrm{cm}s^{-1}$)

**Question:**

What is the phase difference between two waves of wavelength 12cm when one leads with other by (i) 6cm (ii) 9cm (iii) 3cm (iv) 12cm (iv) 36cm (v) 39cm
(In all above, use $\phi = \frac{2\pi x}{\lambda}$)

2. The displacement of a particle on a progressive wave is $y = 2\sin 2\pi(0.25x - 100t)$. Where x and y are in cm and t is in seconds. Calculate (i) wavelength (λ = 0.04m)
(ii) Velocity of propagation of wave. (v = 4ms$^{-1}$)
(Compare with the general equation of a progressive wave)

3. The displacement of a given wave traveling in the X – direction at the time t is $y = a\sin 2\pi\left(\frac{t}{10} - \frac{x}{2}\right)$ m.
Find the; (i) velocity of the wave
(ii) Period of the wave.

4. (a) Find the speed of a compression wave in an iron rod of density 7.7x10$^{3}$kgm$^{-3}$ and whose young's modulus is 2x10$^{11}$Pa. (Use $v = \left(\frac{Y}{\rho}\right)^{\frac{1}{2}}$). (v = 5096.5ms$^{-1}$)
(b) Show that the speed of sound in a gas of molecular mass, M at a temperature of Tkelvins is given by$v = \sqrt{\frac{\gamma RT}{M}}$.
Where γ is the ratio of molar heat capacities.

5. A certain string has linear mass density of 0.25kgm$^{-1}$ and is stretched with tension of 25N. One end is given a sinusoidal motion with frequency 5Hz and amplitude 0.01m. At the time t = 0, the end has zero displacement and is moving in the positive y – direction.
a) Find the wave speed, angular frequency, period, wavelength and wave number.
b) Write a wave function describing the wave.
c) Find the position of the point at x = 0.5m at the time t = 0.1s.

6. A transverse wave of amplitude 10cm and wave length 200cm travels from left to right along a stretched string with speed of 100cms$^{-1}$. Take the origin to be at the left end of the un disturbed string. At a time t=0, the left end of the string is at the origin and moving down wards. Find the;
(i) angular frequency of the wave.
(ii) equation of the wave.
(iii) equation of motion of the left end of the string.
(iv) equation of motion of the particle at 150cm to the right of the origin.
(v) magnitude of the maximum transverse velocity of any particle of the string.
[Ans: (i)$\mathrm{w = \pi rads^{-1}}$(ii) $\mathrm{y - 10sin\pi}\left(\mathrm{t} - \frac{\mathrm{x}}{100}\right)$ cm
(iii)$\mathrm{y = -10sin\pi t}$. (iv) $\mathrm{y = -10sin\pi(t - 1.5)}$ cm.
(v)$\mathrm{v_{max} = 10\pi\ cms^{-1}}$]

**Also see UNEB 1999 No. 4(b), 2000, 2001 No. 3(a)**

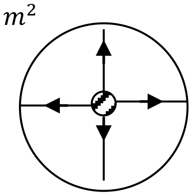
**Transmission of energy by a wave.**

In all progressive waves, energy propagates through the medium in the direction in which the wave travels.
Each particle of the medium has energy of vibration and passes energy onto the next particle.
In simple harmonic motion, where there is no damping, the energy of vibrating particle changes form kinetic energy to potential energy and back, with the total energy, E, remaining constant.
$E_{max}$ = maximum kinetic energy
$E_{max} = \frac{1}{2}mv_{max}^2$ ; $But\ v_{max} = \omega A$
Where A is the amplitude of vibration. Thus;
$E_{max} = \frac{1}{2}mv_{max}^2 = \frac{1}{2}m(\omega A)^2 = \frac{1}{2}m\omega^2 A^2$
Also; $\omega = 2\pi f$ ; where f is the frequency of vibration.
.$\boldsymbol{E_{max}} = \frac{1}{2}m(2\pi f)^2 A^2 = \boldsymbol{2m\pi^2 f^2 A^2}$
As a wave passes through a medium, the energy per unit volume of the medium is the energy per particle times the number of particles, **n** per unit volume. Therefore the energy per unit volume.
$$\frac{\text{Energy}}{\text{Volume}} = \frac{\text{E}_{\text{max}}}{\text{V}} = \frac{2\text{m}\pi^2\text{f}^2\text{A}^2}{\text{V}} = 2\left(\frac{\text{m}}{\text{V}}\right)\pi^2\text{f}^2\text{A}^2 = \boldsymbol{2\rho\pi^2\text{f}^2\text{A}^2}$$
Where $\rho = \frac{mass}{Volume}$ = density of the wave particles.

**Intensity (I)** of the wave is the energy transfer per unit time per unit area perpendicular to the direction of propagation of the waves.
$$\text{Intensity, I} = \frac{\text{Power}}{\text{area}} = \frac{\text{E}_{\text{max}}}{\text{area} \times \text{time}} = \frac{2\text{m}\pi^2\text{f}^2\text{A}^2}{\text{at}}$$
But, mass = density × Volume ⟺ $m = \rho V = \rho(al) = \rho al$.
Where ***a*** is cross section area, and *l* is length.
$$\text{Intensity, I} = \frac{2(\rho al)\pi^2\text{f}^2\text{A}^2}{\text{at}} = 2\left(\frac{l}{t}\right)\rho\pi^2\text{f}^2\text{A}^2 = 2v\rho\pi^2 f^2 A^2$$
$$\mathbf{I} = \boldsymbol{2v\rho\pi^2 f^2 A^2}$$

**Example**
The speed of sound in air is 330ms$^{-1}$. A source of sound of frequency 300Hz radiates energy in all directions at a rate of 10W. Find (i) the intensity of sound at a distance of 20m from the source.
(ii) The amplitude of sound wave at this distance.
(Density of air at s.t.p = 1.29kgm$^{-2}$)

**Solution**
Power, P = 10W, speed=330ms$^{-1}$, f=300Hz, $l$=$r$ =20m
$$\text{Intensity, I} = \frac{\text{Power}}{\text{area}}$$
$But, Area, a = 4\eth r^2 = 4\pi(20)^2 = 1600\pi\ m^2$

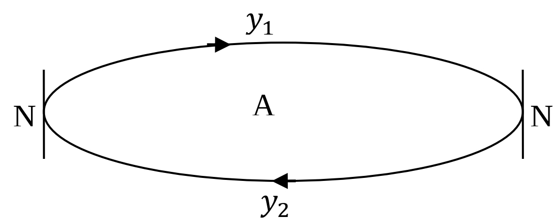


$$\text{Intensity, I} = \frac{10}{1600\pi} = \frac{1}{160\pi} Wm^{-2}$$
(ii)
$$\text{Intensity, I} = 2v\rho\pi^2 f^2 A^2$$
$$\frac{1}{160\pi} = 2(330)(1.29)\pi^2(300)^2 A^2$$
$$A^2 = \frac{1}{160\pi \times 2(330)(1.29)\pi^2(300)^2}$$
$$A = \sqrt{[1.621 \times 10^{-6}]}$$
$$A = 2.63 \times 10^{-12}\text{m}$$

**Principle of Superposition of waves**
The resultant displacement at any point is the sum of the separate displacement s due to the two waves.
Let $y_1$ and $y_2$ represent the displacement of the individual waves in the planes.

When a crest falls on a crest or a trough falls on a trough, the resultant amplitude is double the amplitude of one wave.
When a crest combines with a trough, the resultant amplitude is zero.

**Stationary or standing waves**
Stationary waves are due to superposition of two progressive waves having the same speed and frequency and nearly equal amplitude but traveling in opposite directions.
Consider two waves where $y_1$ and $y_2$,




where $y_1 = a\sin(\omega t - kx)$. Hence $y_2 = a\sin(\omega t + kx)$
From super position principle, The resultant displacement y is;
$$y = y_1 + y_2$$
$$y = a\sin(\omega t - kx) + a\sin(\omega t + kx)$$
$$y = 2a\sin(\omega t)\cos(kx)$$
$$y = (2a\cos kx)\sin\omega t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(i)$$

Compare this equation with the general progressive wave equation
$$y = (A)\sin\omega t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(ii)$$

⟺ $\boldsymbol{Amplitude. A = 2a\cos kx}$
The amplitude A is maximum and is equal to **2a** at $x$ = 0, $x$ = **λ/2**, $x$ = **λ** and so on.
These points are thus **antinodes** and hence separation between consecutive antinodes on a stationary wave is **λ/2**.
The displacement is zero when ***x* = λ/4, *x* = 3λ/4, *x* = 5λ/4.** and so on. These points are called **nodes** and hence they are midway between consecutive antinodes.

**Note*:*** Within a stationary wave, there is no flow of energy through a medium.There is energy of motion between each vibrating segment but this energy is not transferred across the node and is stationary.

***Difference between progressive and stationary waves***

| ***Stationary wave*** | ***progressive*** |
|---|---|
| -All particles between successive nodes, have their vibration are in phase. | -The phase of vibration of points near each other are all different |
| -Each points along the wave has a different amplitude i.e. Amplitude = 2acosθ | -All points along the wave vibrate with the same amplitude. |
| -Do not transfer energy. | -Transfer energy from one point to another. |
| -The wave profile does not move along the medium | - The wave profile moves along the medium with the wave speed. |
| -The medium doesn't move. | -The medium moves. |

❖ ***<u>Transverse Stationary wave in strings.</u>***

***Modes of vibration***

The ends of a stretched string are fixed and there fore the ends of the string must be the displacement nodes.

If the string is displaced in the middle, a stationary wave is formed.

**<u>Fundamental note:</u>**

- Is a note with the lowest audible frequency.
- It is the note produced at the first position of resonance.

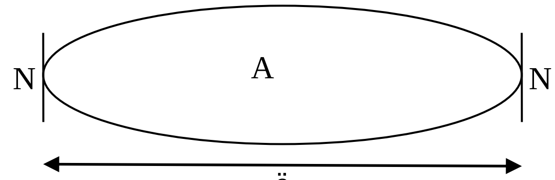
**<u>Overtones:</u>**

- Is a note whose frequency is higher than the fundamental frequency.

**<u>Harmonics:</u>**

- Is a note whose frequency is an integral multiple of of the fundamental frequency.

***(i) First Harmonic (fundamental note)***




$l = \frac{ë}{2}$

The wave formed in this case is the simplest form of vibration and is called the fundamental .

The frequency at which it vibrates is called the fundamental frequency.

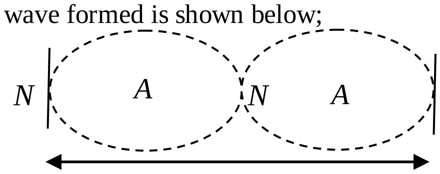
If f is the frequency (Fundamental frequency). Then

$f_1 = \left(\frac{v}{\lambda}\right)$, But $\lambda = 2l$

$f_1 = \left(\frac{v}{2l}\right)$, Where v is the speed of the wave.

***(ii) Second Harmonic (first Overtone).***

When the wave is plucked quarter way from one end, the wave formed is shown below;

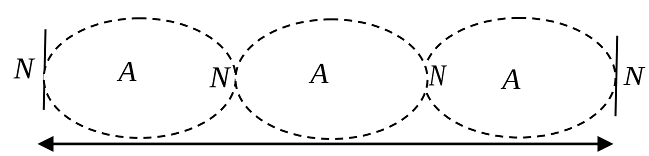



$l = \lambda$

If $f_2$ is the frequency of the wave,then;

$$f_2 = \frac{v}{\lambda_2} = \frac{v}{l} = \frac{2}{2} \times \frac{v}{l} = 2 \times \left(\frac{v}{2l}\right) = 2f_1$$

(iii) ***Third harmonic (2nd overstone)***




$l = 3 \times \frac{\lambda}{2}$

$$f_3 = \frac{v}{\lambda_3} = \frac{v}{\left(\frac{2}{3}l\right)} = 3 \times \frac{v}{2l} = 3f_1$$

Frequencies which are higher than fundamental frequency are called overtones.

Waves formed by a stretched string are of the frequencies $f_{1,} 2f_{1,} 3f_1, 4f_1 ..$

**In general**

If $\frac{\lambda_n}{2} = \frac{l}{n} \Leftrightarrow \lambda_n = \frac{2l}{n}$, The frequency $f_n$ of the n[th] harmonic is given by:-

$$f_n = \frac{v}{\lambda n} = \frac{v}{\left(\frac{2l}{n}\right)} = n\left(\frac{v}{2l}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

But when; $n = 1, f_1 = \frac{v}{2l}$

Hence; $\boldsymbol{f_n = n\left(\frac{v}{2l}\right) = nf_1}$

The frequency of the various overtones is whole number multiples of the fundamental frequency.

Recall, the speed of the vibrating string is;

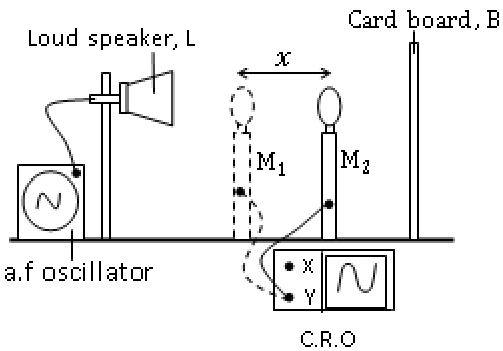
$v = \sqrt{\frac{T}{\mu}}$……………………………………(ii)

Where $\boldsymbol{T}$ is the tension in string and $\boldsymbol{\mu}$ is the mass per unit length.

From eqn. (i) and (ii):

$f_n = n\left(\frac{v}{2l}\right) = \frac{n}{2l}\sqrt{\frac{T}{\mu}}$; where n = 1, 2, 3, 4, ……….

**Experiment to determine the speed of sound in air using an a.f oscillator and a C.R.O.**




-A loud speaker, L is connected to an oscillator of constant frequency, f.

-A microphone, M is connected to the Y- plate of the C.R.O and placed between the speaker and a card board, B.

-The microphone, M is moved away from the cardboard, B towards the loud speaker, L until the amplitude of the wave oscillation observed from the C.R.O is maximum. This position is noted as $M_1$.

-This is the position of an antinode of the stationary wave formed by superposition of the incident and reflected progressive waves.

-The microphone, M is moved further towards the loud speaker, L until the amplitude of the wave oscillation observed from the C.R.O increases to a maximum again. This position is noted as $M_2$.

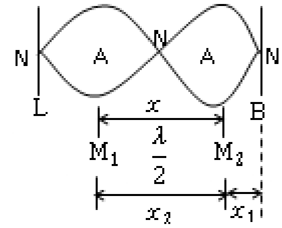
-The distance between the microphone positions $M_1M_2 = x$ between successive maximums is measured.

-The velocity of sound in air is thus calculated from;

$v = f\text{ë}$

$v = 2fx$

**Theory:**




$x = x_2 - x_1$

$\frac{\text{ë}}{2} = x_2 - x_1$

$\text{ë} = 2(x_2 - x_1)$

$\text{ë} = 2x$

From the wave equation;

$v = f\text{ë}$

$v = 2fx$

$\boldsymbol{v = 2f(x_2 - x_1)}$

***Examples***

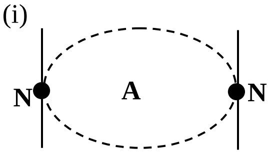
1. A wire of length 400mm and mass 1.2gm is under a tension of 120N.

(i) What is the fundamental frequency of vibration?

(ii) The frequency of the 3[rd] harmonic

**Solution**

(i)




$$l = \frac{\text{ë}}{2}$$

$\text{ë}_1 = 2l = 2(400 \times 10^{-3})$

$\text{ë}_1 = 0.8$ m

$v = f_1\text{ë}_1 = \sqrt{\frac{T}{\text{ì}}}$

$0.8f_1 = \sqrt{\frac{120}{\left(\frac{1.2\times10^{-3}}{400\times10^{-3}}\right)}}$

$\boldsymbol{f_1 = 250}$ **Hz**

Alternatively;

$$f_1 = \frac{1}{2l}\sqrt{\frac{T}{\text{ì}}} = \frac{1}{2(400\times10^{-3})}\sqrt{\left\{\frac{120}{\left(\frac{1.2\times10^{-3}}{400\times10^{-3}}\right)}\right\}}$$

$f_1 \quad = 250$ Hz

(ii)

$$\boldsymbol{f_3} = 3f_1 = 3 \times 250 = \mathbf{750}\text{Hz}$$

2. The mass of the vibrating length of sonometer wire is 1.20g and it is found that a note of frequency 512 Hz is produced when wire is sounding its second overtone .If the tension of the wire is 100N, calculate the vibrating length of wire.

**Solution:**

$$f_1 = \frac{1}{2l}\sqrt{\frac{T}{\text{ì}}} = \frac{1}{2l}\sqrt{\left\{\frac{100}{\left(\frac{1.2\times10^{-3}}{l}\right)}\right\}}$$

$f_3 = 3f_1$

$$512 = 3 \times \frac{1}{2l}\sqrt{\left\{\frac{100l}{1.2\times10^{-3}}\right\}}$$

$\boldsymbol{l = 0.715}$ **m**

***Question***

A plane string 1.5m long is made of steel of density $7.7\times10^{3}kgm^{-3}$ and Young's modulus $2\times10^{11}NM^{-2}$.
It is maintained at a tension which produces an elastic strain of 1% in the string. What is the fundamental frequency of the transverse vibration of the string.

**Experimental verification of** $f_1 = \frac{1}{2l}\times\sqrt{\frac{T}{\mu}}$

These frequency $f_1$of the fundamental mode of vibration is given by $\frac{1}{2l}\times\sqrt{\frac{T}{\mu}}$ **, it** follows that;

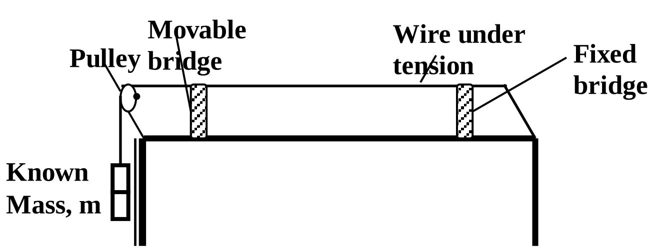
**(i)** $f_1 \propto \frac{1}{l}$ (T and μ constant)

**(ii)** $f_1 \propto \sqrt{T}$ ( $l$ and μ constant)

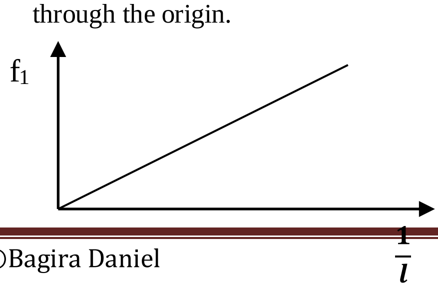
**(iii)** $f_1 = \sqrt{\frac{1}{\mu}}$ (T and $l$ constant)

These relationships are sometimes referred to as the laws of vibration of a stretched string. They may be verified experimentally by using a**sonometer**described below:

i) $f_1 \propto \frac{1}{l}$

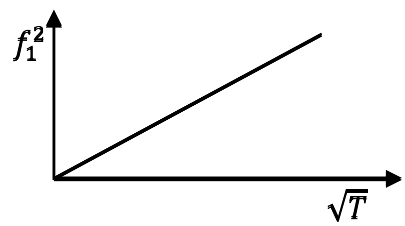



- A rider is placed at the centre of the wire and a vibrating tuning fork of known frequency held against one of the bridges.
- The length of the wire between the bridges is adjusted until the rider falls off the wire.
- The procedure is repeated using different tuning forks of known frequencies without altering T and μ.
- The result are tabulated including values of $f_1$ and $\frac{1}{l}$
- A graph of $f_1$ against$\frac{1}{l}$ is plotted and is linear passing through the origin.




**(ii)** $f_1 \propto \sqrt{T}$

- A rider is placed at the centre of the wire and a vibrating tuning fork of known frequency held against one of the bridges.
- The tension in the wire is adjusted by adding known weights (or masses) on the scale pan until the rider falls off the wire. Measure and record the masses, m added.
- The procedure is repeated using different tuning forks of known frequencies without altering $l$ and μ.
- The results are tabulated including values of $f_1^2$and m or T.
- A graph of $f_1^2$ against $m\ or\ \sqrt{T}$is plotted and is linear passing through the origin.



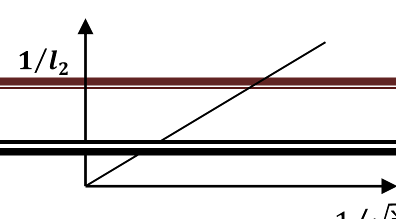

Hence $f_1 \propto \sqrt{T}$

**Note;**

This experiment can also be described to show that a wire under tension can vibrate with more than one frequency.
This is done by keeping $l$ and T constant and plucking the wire at distances of $\frac{1}{2}l\ ,\frac{1}{4}l\ and\ \frac{1}{8}l.$
Different tuning forks are held close to the vibrating wire in each case to find which one resonates with it.

**(iii)** $f_1 \propto \frac{1}{\sqrt{\mu}}$

- The mass per unit length μ is determined by weighing different wires.
- One such wire of known μ is connected across fixed bridges and sent into vibrations by plucking it at its centre.
- A second wire $l_2$ is connected on the sonometer box across the movable bridges and a rider placed at its centre.
- The length $l_2$ is adjusted until the rider falls off the wire.
- The procedure is repeated with wires of different masses per unit length.
- The results are tabulated including values of $\frac{1}{l_2}$and **1**/√ì .
- A graph of$\frac{1}{l_2}$ against**1**/√ì is plotted and is linear through the origin.



- The graph shows that $\frac{1}{l} \propto \frac{1}{\sqrt{T}}$ and since $f_1 \propto \frac{1}{l}$. Hence $f_1 \propto \frac{1}{\sqrt{T}}$.

## ❖ Longitudinal stationary waves in pipes

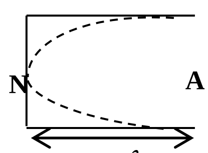
***(a) Closed pipes.***

This consist essentially of a metal pipes closed at one end and open at the other.

*Closed pipes boundary conditions.*

At the closed end, there is a displaced node.
At the open end here is displaced antinode.
The allowed oscillation modes or standing wave patter are:-

(i) ***Fundamental note***

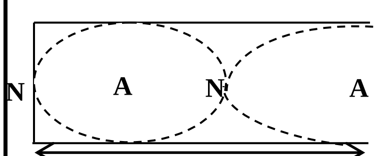



$$l = \frac{\lambda_1}{4}$$

Fundamental frequency, $f_1 = \frac{v}{\lambda_1} = \frac{v}{4l}$ ………………(i)

Fundamental or lowest audible frequency ($f_1$)
It is obtained when the simplest stationary wave form is obtained.

(ii) ***First overtone*** (3rd harmonic)

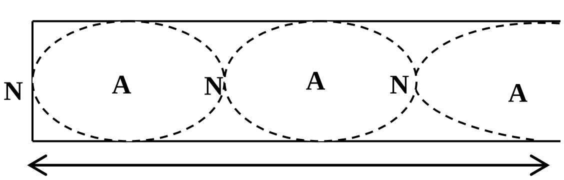



$$l = \frac{3}{4}\lambda_3$$

Frequency of first overtone $f_3$ is given by;

$$f_3 = \frac{v}{\lambda_3} = \frac{v}{\left(\frac{4l}{3}\right)} = \frac{3v}{4l} = 3 \times \left(\frac{v}{4l}\right) = 3f_1$$

(iii) ***Second overtone*** (5th harmonic)




$$l = \frac{5\lambda_5}{4}$$

$$f_5 = \frac{v}{\lambda_5} = \frac{v}{\left(\frac{4l}{5}\right)} = \frac{5v}{4l} = 5f_1$$

The frequencies obtained with a closed pipe are $f_1$, $3f_1$, $5f_1$, $7f_1$ $9f_1$, etc i.e. only odd harmonics' are obtainable.

**In general;** $f_n = n\left(\frac{v}{4l}\right) = nf_1$, Where, **n** = 1, 3, 5, 7………

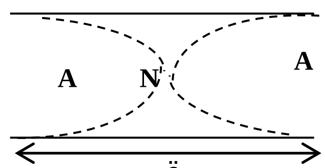
***(b) Open pipes***

These are Pipes which are open at both ends.

*Open pipes boundary conditions*

Antinodes are at both ends.
The allowed oscillation modes or standing wave patter are:-

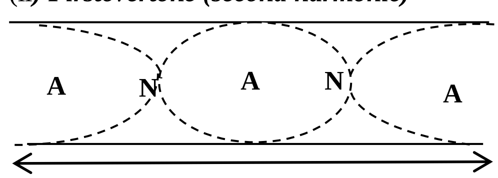
(i)***Fundamentalnote.***




$$l = \frac{\lambda_1}{2}$$

Fundamental frequency; $f_1 = \frac{v}{\lambda_1} = \frac{v}{2l}$

(ii***) Firstovertone (second harmonic)***




$$l = \lambda_2$$

$$f_2 = \frac{v}{\lambda_2} = \frac{v}{l} = \frac{2}{2} \times \frac{v}{l} = 2 \times \left(\frac{v}{2l}\right)$$

$$f_2 = 2f_1$$

Thus frequencies for notes produced by open pipes are $f_1, 2f_1, 3f_1, 4f_1$………..

**In general;** $f_n = n\left(\frac{v}{2l}\right) = nf_1$, Where, **n** = 1, 2, 3, 4………

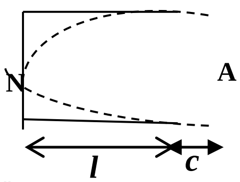
So an open pipe can produce both odd and even harmonics.
Therefore, open pipes produce a richer note than that from a similar closed pipe, due to the extra harmonics.

**End correction**

Then, at the open end of the pipe is free to move and hence the vibration at this end of the sounding pipe extend a little into the air outside.

Antinodes of the stationary were due to any note is in practice a distance, c from the open end. The distance, c is known as the end correction.

*For the closed pipe*;-
Fundamental mode




$\frac{\ddot{e}_1}{4} = l + c \Leftrightarrow \ddot{e}_1 = 4(l + c)$

$f_1 = \frac{v}{\ddot{e}_1} = \frac{v}{4(l + c)}$

Fundamental frequency,

$\boldsymbol{f_1 = \frac{v}{4(l + c)}}$

*For open pipe*;-
Fundamental mode,

$\frac{\ddot{e}_1}{2} = l + 2c \Leftrightarrow \ddot{e}_1 = 4(l + 2c)$

$f_1 = \frac{v}{\ddot{e}_1} = \frac{v}{2(l + 2c)}$

Fundamental frequency,

$\boldsymbol{f_1 = \frac{v}{2(l + 2c)}}$

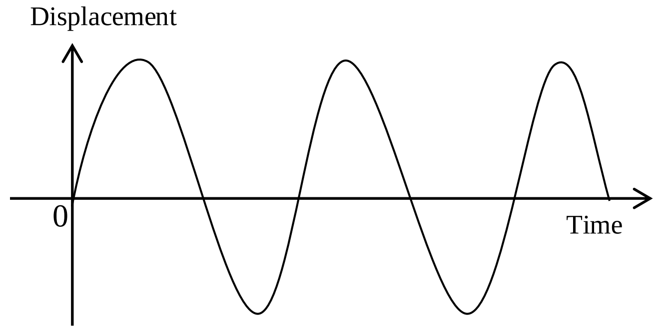
## OSCILLATIONS:

An oscillation is a complete to and fro motion of the particles about an equilibrium position.
In oscillations, the displacement from a point and speed vary with time.
Mechanical oscillations are oscillations in which the energy of vibrating particles is passed and stored by vibrating particles.
Electrical oscillations are oscillations in which the energy is passed and stored in the electrical and magnetic fields.

Types of Oscillations.

(a) Free Oscillations;

These are oscillations which occur in absence of dissipative forces.
The energy and amplitude of oscillation remain constant with time and the system oscillates indefinitely.
Dissipative forces.are forces which can cause loss of energy from the system.
Examples; Friction, Air resistance, viscosity



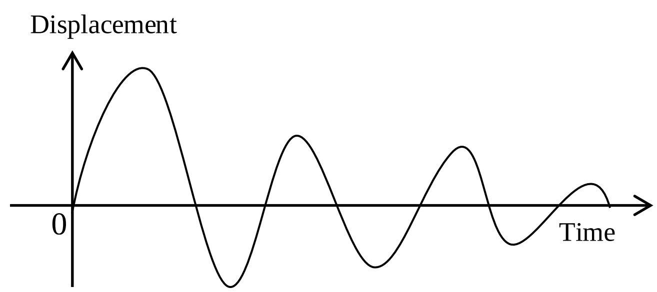

(b) Damped Oscillations:

These are oscillations which occur in presence of dissipative forces.
The system loses energy to the surroundings and theamplitude of oscillation decreases gradually until the system finally comes to rest.
Damping is the continuous decrease in the amplitude of oscillations due to dissipation of energy of a body or system. It is caused by external resistive forces.

Types of damped oscillations

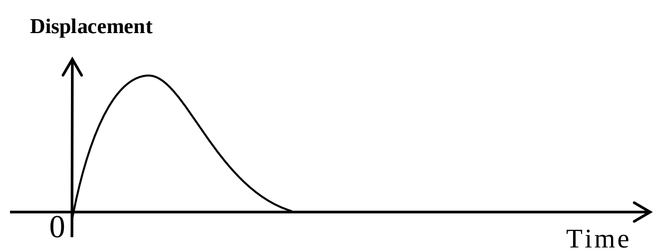
(i) Under damped oscillations

These are oscillations in which the system experiences low resistive/ dissipative forces such that it loses energy gradually.
The amplitude of the oscillations decreases gradually until the system finally comes to rest.




(ii) Critically damped oscillations

These are oscillations which occur when a system is displaced but doesn’t oscillate and returns to the equilibrium position in the minimum time possible.
The magnitude of the resistive forces is such that they don’t allow the system to vibrate past its equilibrium position.




Critical damping is applied in;

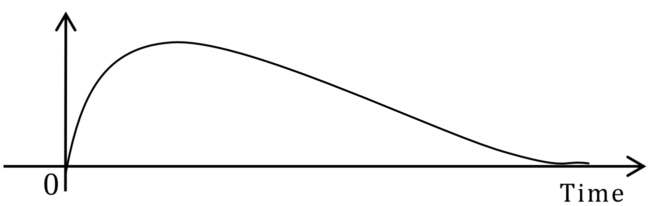
- Shock absorbers
- Office doors

- Moving coil instruments such as ammeters, voltmeters and galvanometers to bring the pointers to rest instantly so that the reading can be taken.

(iii) Over damped oscillations.

These are oscillations which occur when a system is displaced and doesn't oscillate but slowly returns to the equilibrium position.

The magnitude of the resistive forces is such that they don't allow the system to vibrate past its equilibrium position but it takes a long time to return to the equilibrium position.




**Differences between Free and Damped oscillations**

| Free oscillations | Damped oscillations |
|---|---|
| -Wave energy remains constant. | -Wave energy reduces with time. |
| -Amplitude of oscillation remains constant. | -Amplitude reduces with time. |
| -Occurs in absence of dissipative forces. | -Occurs in presence of dissipative forces. |
| -System oscillates indefinitely. | -System finally comes to rest. |

(c) Forced Oscillations:

These are oscillations in which the system subjected to some degree of damping is made to continuously oscillate by subjecting to an external periodic force.

The external periodic force serves to restore the energy lost by the system as a result of dissipative forces.

The Forcing Frequency is the frequency of the external periodic force.

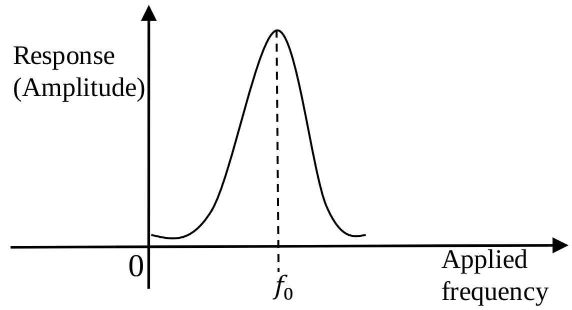
**Resonance in pipes.**

Any forced oscillating system (air column, mechanical system, diving board) gives a maximum response when the diving frequency f, is equal to the natural frequency $f_o$of the forced system.

The system absorbs maximum energy from the external period force and vibrates with maximum amplitude. The system is saidtoresonate when this happens.

**Resonance**: Is the vibration or oscillation of a system at its natural frequency with maximum energy and amplitude due to vibrations received from one point an external source of the same frequency.

***Sketch of response against frequency.***




Resonance occurs when a particular body or system is set into oscillation at its own natural frequency as a result of impulses received from other systems vibrating with the same frequency.

If the prongs of a tuning fork are held over the top of the pipe , air inside is set in vibration by the periodic force extended on it by the prongs .The vibration are feeble as they are forced vibrations and the intensity of the sound is correspondingly small. But when a tuning fork of the same frequency as the fundamental frequency of the pipe is held over it , the air inside is set into resonance by periodic force and the amplitude of vibration is loud . A loud note of the same frequency as the note is heard coming from the pipe.

In general, for a tube of varying length, resonance is obtained for some lengths where a stationary wave is set up with an antinode at one end and node at the closed end.

## ② SOUND WAVES

Sound is longitudinal wave. It therefore consists of compressions and rare factions.

Sound waves are generated when particles of a medium are set into oscillation by a vibrating object. The vibrating object superposes an oscillatory to and fro motion on the particles of the transmitting medium along the direction of the waves.

Sound exhibits all the properties of waves except polarization which is not exhibited by longitudinal waves.

In addition, sound is a mechanical wave; hence it requires a material medium for propagation. I.e. it cannot travel through a vacuum.

Properties of sound

Sound exhibits properties of waves including:

- Reflection
- Refraction
- Interference
- Diffraction

**Reflection:**

Sound waves can be reflected by obstacles obeying the laws of reflection.

An echo is the reflected sound from an obstacle.

The time that elapses between the original sound and the echo determines whether the observer distinguishes the original sound from the echo or not.

This time depends on the:

(i) Distance of observer from the reflector or obstacle.

(ii) Speed of sound

If the time is less than 0.1seconds, the human ear cannot distinguish between the original sound and the echo. If the time is about0.1 seconds, the original sound mixes up with the echo and sound is heard appears to be prolonged. This effect is called **reverberation.**
The time taken for the intensity of sound to completely die away is called <u>Reverberation time.</u>

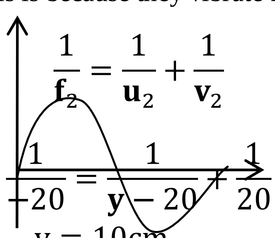
**<u>Implications of reverberation in a hall.</u>**

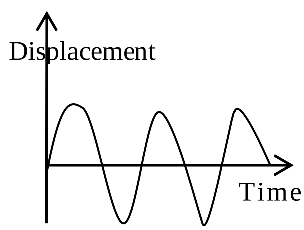
(i) In large halls soft clothings, cushions and human skin absorb sound thus reducing its intensity.
This causes the music or speech to become weaker and inaudible. In such cases, reverberation can improve audibility of sound.
(ii) Excessive reverberation however makes the speech or music indistinct and confused.

**<u>Musical Notes:</u>**
A musical note or tone is a sound of regular frequency.
Music is a combination of such sounds.
Noise on the other hand is sound from sources with irregular frequencies.

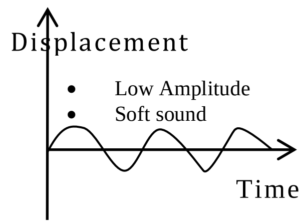
<u>Characteristics of musical notes or musical sounds</u>

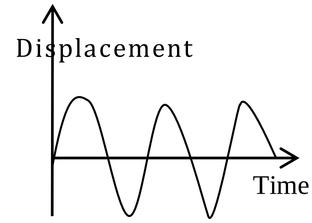
**(i) Pitch**
This is the highness or lowness of sound.
It is a x-tic of sound which enables one to distinguish a high note from a low note.
It depends on the frequency of vibrations. The higher the frequency the higher the pitch of sound.
Small and short vibrating objects produce high pitched sound. This is because they vibrate faster.




**(ii) Loudness(volume of sound)**
Loudness refers to the amount of sound energy entering the ear per second.
It depends on the intensity and hence amplitudeof sound vibrations (since; $\boldsymbol{Intensity} \propto (\boldsymbol{Amplitude})^2 \;\Leftrightarrow \boldsymbol{I} \propto \boldsymbol{A^2}$.)
Large objects produce loud sound)
Large objects produce loud sound. This is because a large volume of air is set into vibrations.




**Intensity of sound.**
This is the rate of flow of energy through an area of one square metre perpendicular (or normal) to the path of travel of sound.
The S.I unit of intensity is $Wm^{-2}$.
Intensity of sound is directly proportional to the square of amplitude of sound.i.e. $\boldsymbol{I} \propto \boldsymbol{A^2}$;
Where I =intensity of sound and A= amplitude.

From the definition of intensity of sound, the intensity, **I** at any point at a distance, **r** from the source is given by;

$$\boldsymbol{Intensity} = \frac{\boldsymbol{Power}}{\boldsymbol{Area}}$$

$$\boldsymbol{I} = \frac{\boldsymbol{P}}{\boldsymbol{A}} \Leftrightarrow \boldsymbol{I} = \frac{\boldsymbol{P}}{\boldsymbol{4\pi r^2}} \Leftrightarrow \boldsymbol{I} = \left(\frac{\boldsymbol{P}}{\boldsymbol{4\pi}}\right)\frac{\boldsymbol{1}}{\boldsymbol{r^2}}$$

But $\frac{P}{4\pi}$is a constant, hence $\frac{P}{4\pi} = \boldsymbol{k}$

$$\Leftrightarrow \boldsymbol{I} = \boldsymbol{k}\left(\frac{\boldsymbol{1}}{\boldsymbol{r^2}}\right)$$

$$\Leftrightarrow \boldsymbol{I} \propto \left(\frac{\boldsymbol{1}}{\boldsymbol{r^2}}\right)$$

Therefore, as the distance from the source of a sound wave increases, the intensity of sound decreases following the inverse square law.
This explains why the loudness of sound decreases with increasing distance from the source.

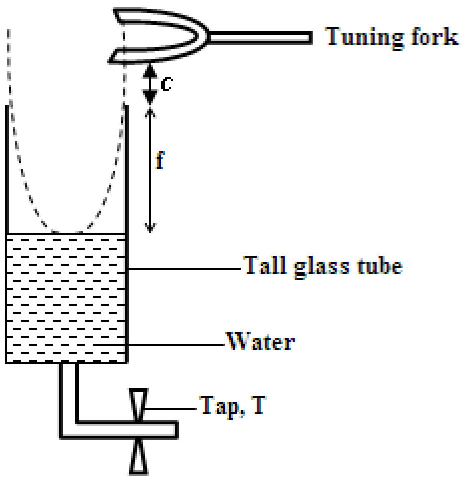
<u>Causes of the decrease in loudness of sound with increasing distance from the source.</u>

- Loss of energy of the wave to the transmitting medium.
- Wave energy spreads over a wider area for a point at a distance, r from the source.

**(iii) Quality(Timbre)**
This is the characteristics of a note that distinguishes it from another note of the same pitch and loudness.
It depends on the intensity of harmonics (overtones) present in the note.
Notes that contain many harmonics are usually of better quality than notes with fewer harmonics.

**<u>Measurement of velocity of sound in air using a resonance tube.</u>**
A sounding tuning fork is held over the open end of a tube T filled with water.

The level of water in the tube is gradually lowered until aloud sound is heard (resonance is obtained) at the some position. The first two resonance lengths are obtained and the corresponding vibrating air column lengths $l_1$ and $l_2$ are measured.




At firstresonance, $l = l_1$.

Hence $\frac{\lambda}{4} = l_1 + c$ ……………….. (i)

At second resonance, $l = l_2$.

Hence $\frac{3\lambda}{4} = l_2 + c$ …………….. (ii)

From equation (i) and (ii), $\lambda = 2(l_2 - l_1)$

From $v = f\lambda$

$\Rightarrow v = 2f(l_2 - l_2)$

Hence $\underline{v = 2f(l_2 - l_1)}$

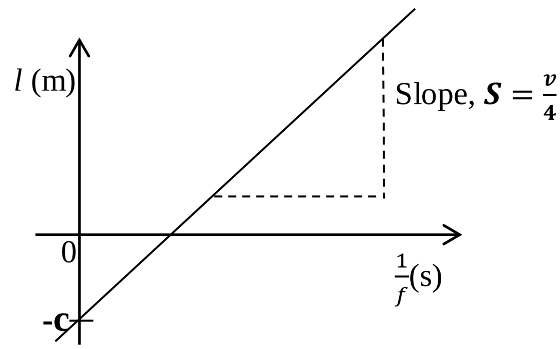
**Alternatively:**

Different tuning forks of different known frequencies usedand the length of air column is adjusted (by opening the tap T) until the first loud sound is heard.

The lengths $l$ of the vibrating air column is measured and the results are tabulated including values of $\frac{1}{f}$.

| Frequency,f(Hz) | Length, $l$(m) | $\frac{1}{f}$ ($Hz^{-1}$) or (s) |
|---|---|---|
| - | - | - |
| - | - | - |

A graph of $l$ against $\frac{1}{f}$ is plotted and its slope S calculated.




- The speed of sound in air is calculated from; $v = 4S$
- The end correction is equal to the intercept on the $l$-axis.

<u>Theory of Experiment;</u>

At resonance; $l + c = \frac{ë}{4}$ …..……………….(i)

But, Velocity; $v = fë$ …………………...(ii)

From Eqn.(i) and Eqn.(ii);

$l + c = \frac{v}{4f} \Leftrightarrow l = \frac{v}{4f} - c \Leftrightarrow l = \left(\frac{v}{4}\right)\frac{1}{f} - c$

Then, comparing the expression;

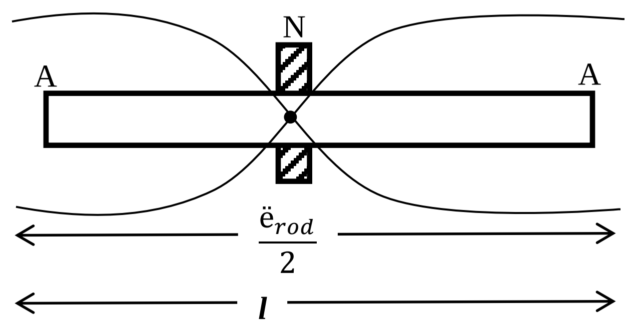
$\boldsymbol{l = \left(\frac{v}{4}\right)\frac{1}{f} - c}$ **,** withthe equation of a straight line;

$\boldsymbol{y = mx + c.}$

It implies that, the slope,**S**=**m** $=\frac{v}{4}$ and the $l$- intercept, $\boldsymbol{c = -c}$

**Note:** For a pie of radius r, the end correction of the pipe is given by; $\mathbf{c = 0.6r}$.

## <u>WAVES IN RODS AND SPEED OF SOUND IN RODS.</u>




If the rod AA of length $l$ is stroked along its length by a rosined cloth, a stationary longitudinal wave is set up in the rod due to reflections at its ends.

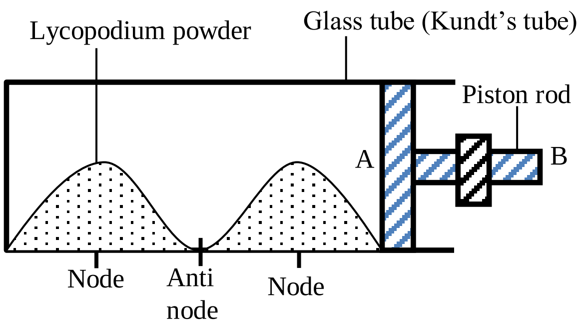
Since the mid-point of the rod is fixed, this is the node (N) of the wave formed and since the ends of the rod are free, these are displacement antinodes (A).

Thus, $\boldsymbol{l = \frac{ë_{rod}}{2}} \Leftrightarrow ë_{rod} = 2l$. Whereë$_{rod}$ is the wave length of sound in the rod.

The velocity of sound in the rod is obtained from;

$$\boldsymbol{v_{rod} = fë_{rod} = 2fl}$$

**<u>Measurement of velocity of sound in air using a Kundt'stube.</u>**




The length $\boldsymbol{l_{AB}}$ of the rod AA is measured and recorded.
The rod is clamped at its mid-point and placed inside the glass tube such that it covers one side of the tube.

The rod is then stroked at B and it vibrates longitudinally causing the lycopodium powder to be violently agitated.

The powder moves away from the anti-nodes where air is vibrating strongest and settlesinto definite heaps at the nodes which are positions of permanent rest.
The length $\boldsymbol{l_{NN}}$between consecutive heaps (nodes) is measured.

☞ Speed of sound in air

$$l_{NN} = \frac{\ddot{e}_{air}}{2} \Leftrightarrow \ddot{e}_{air} = 2l_{NN}$$

$$\boldsymbol{v_{air} = f\ddot{e}_{air} = 2fl_{NN}} \text{………….. (i)}$$

☞ Speed of sound in the rod

$$l_{AB} = \frac{\ddot{e}_{AB}}{2} \Leftrightarrow \ddot{e}_{AB} = 2l_{AB}$$

$$\boldsymbol{v_{rod} = f\ddot{e}_{AB} = 2fl_{AB}} \text{………… (ii)}$$

Where f is the frequency of a note from the rod.
Then from Equations (i) and (ii);

$$\boldsymbol{v_{rod} = \left(\frac{l_{AB}}{l_{NN}}\right) v_{air}}$$

**Examples**

**1** A progressive and stationary wave each have a frequency of 240Hz and a speed of 80ms$^{-1}$

**(i)** Calculate the phase difference between two vibrating points in the progressive wave when they are 6 cm apart.

**Solution**

$$f = 240\,Hz, \quad v = 80\,ms^{-1};\ \lambda = \frac{\text{v}}{\text{f}} = \frac{80}{240} = \frac{1}{3}\text{m}$$

$$\text{Phase difference, } \varphi = \frac{2\pi x}{\lambda} = \frac{2\pi \times 6\times10^{-2}\times 3}{1}$$

$= 0.36\pi$ rad.

(ii) Distance between nodes in the stationary wave.

$$\lambda = \frac{\text{v}}{\text{f}} = \frac{80}{240} = \frac{1}{3}\text{m}$$

Hence distance between nodes $= \frac{\lambda}{2} = \frac{1}{3}\times\frac{1}{2} = \frac{1}{6}m$

**2.** A plane progressive wave is given by; $y = a\sin\left(100\eth t - \frac{10}{9}\eth x\right)$

(i) Write the equation of a progressive wave which would give rise to the stationary wave if superimposed on the above.
{Ans: $y = a\sin\left(100\eth t + \frac{10}{9}\eth x\right)$}

(ii) Find the equation of the stationary wave and hence determine the amplitude of vibration.
$y_1 = a\sin\left(100\eth t - \frac{10}{9}\eth x\right)$and $y_1 = a\sin\left(100\eth t + \frac{10}{9}\eth x\right)$
Using the principle of superposition, resultant displacement $y = y_1 + y_2$
Hence ;

$$y = a\sin\left(100\eth t - \frac{10}{9}\eth x\right) + a\sin\left(100\eth t + \frac{10}{9}\eth x\right)$$

$$y = a\sin(100\eth t) \times \cos\left(\frac{10}{9}\eth x\right)$$

$$y = 2a\cos\left(\frac{10}{9}\eth x\right)\sin(100\eth t)$$

Hence Amplitude $= A = 2a\cos\left(\frac{10}{9}\eth x\right)$

(i) Determine the velocity and frequency of the stationary wave.

$$\text{f} = \frac{1}{\text{T}} \text{ where } \text{T} = \frac{2\pi}{\omega}$$

$$\text{f} = \frac{\omega}{2\pi} = \frac{100\pi}{2\pi} = 50\,Hz$$

(ii) Velocity of the wave

$$k = \frac{10}{9}\pi = \frac{2\pi}{\lambda}$$

$$\lambda = 1.8mm = 1.8\times10^{-3}\,m$$

$$v = f\lambda = 50\times1.8\times10^{-3} = 0.09\,ms^{-1}$$

**4** A glass tube open at the top is held vertically and filled with water. A tuning fork vibrating at 264 Hz is held above the table and water is allowed to flow out slowly .The first resonance occurs when the water level is 31.5cm from the top while the 2$^{nd}$ resonance occurs when the water level is 96.3cm from the top.
Find the;-
(i) Speed of sound in the air column.

At first resonance, $\frac{\lambda}{4} = l_1 + c$

$$\frac{\lambda}{4} = 0.315 + c\text{………. …….}(i)$$

At second resonance, $\frac{3\lambda}{4} = l_2 + c$

$\frac{3\lambda}{4} = 0.963 + c..........\ ....(ii)$

Equation (ii) – (i) you get $\frac{\lambda}{2} = 0.963 - 0.315$

$\lambda = 2 \times (0.963 - 0.315) = 1.296m$

$v = f\lambda = 264 \times 1.296 = 342.144ms^{-1}$

(ii) End correction.

$\frac{\lambda}{4} = 0.315 + c$

$c = \frac{\lambda}{4} - 0.315 = (\frac{1.296}{4} - 0.315)$

$c = 0.324 - 0.315 = 0.009m$

**5.**A tube of length 1m has its lowest resonance frequency at 86.2Hz. With a tube of identical dimensions but open at both ends. The first resonance position occurs at171Hz.Calculate the; (i) speed of sound (347.64ms$^{-1}$)
(ii) end correction of the tube.(0.00825m)

**Exercise**

1. A stretched wire of length 0.75m, radius 1.36mm and 1380kgm$^{-3}$ is dumped to both sides and is plucked in the middle .The fundamental note is produced by the wire has the same frequency as the first overtone in the pipe of length 0.15m closed at one end .
 (i) Sketch the standing wave pattern in the wire.
 (ii) Calculate the tension in the wire
(V of sound in air =330ms$^{-1}$) $\left[ v = \sqrt{\frac{I}{\mu}} \right]$

2. (a) A wave of amplitude 0.2m, wave length 2m and frequency 50Hz , propagating in the X- direction . If the initial displacement is 0 at point x = 0.
 (i) Write the expression of the displacement of the wave at any time.
 (ii) Find the speed of the wave.

(b) 2 waves of frequency 256Hz respectively travel with a speed of 340m $^{-1}$ in a medium.
(i) Find the phase difference of an amplitude point 2m from where they are initially in phase.
(ii) Describe an experiment to demonstrate that a metal wire under tension can vibrate with more than one frequency.

3. Give the factors that affect the frequency of the transverse wave traveling along a stretched wire and how the frequency varies with each factor.
 (b)A string of strength 31.6cm of fixed at both ends so that it is taut. The lowest frequency of the transverse were it can produce is 880Hz .Calculate the speed of wave.

 (c) A long glass tube is filled with water. A tuning fork is held at the mouth of the tube and the tube is gradually emptied. Explain what happens.

4. A small speaker emitting a note of 250Hz is placed over the open upper end of a vertical tube when it is full of water. When the water is gradually run out of the tube, the air when it is 0.98m below the top column resonates, initially when the water surface is below 0.310 below the top. Find V and end correction.

5. (i) State the conditions that lead to the establishment of the standing wave.
 (ii) A uniform tube 50cm long stands vertically with its lower end dipping into the water. The tube resonates to a tuning fork of frequency 256Hz when its length above water is 12cm and again when it is 39.6cm. Eliminate the lowest freq. to which the tube resonates when it is open at both ends.

 (iii) A wire of mass 1.0x 10$^{-2}$kg and decimeter 6.0x10$^{-4}$m is stretched between rigid supports 1.0m apart.
The tension in the string is 60N. Find the change in the freq. of the fundamental rock when the temperature of the wire is lowered by 100K given that the speed $v = \sqrt{\frac{\iota}{\mu}}$

Young's modulus = 2.0x10$^{11}$pa.
Linear expansion = 1.5x10$^{-6}$K$^{-1.}$

6. The results below were obtained in an experiment to determine the speed of of sound in air using a resonance tube.

| $l(cm)$ | $f(Hz)$ |
|---|---|
| 16.0 | 512.0 |
| 17.5 | 480.0 |
| 20.3 | 426.6 |
| 22.0 | 384.0 |
| 26.5 | 320.0 |

Plot a suitable or appropriate graph and use it to determine the;
 (i) Velocity of sound in air
 (ii) End correction of the tube

7. A closed pie of length 0.68m is blown at its open end so that it produces the 1$^{st}$ harmonic.
(i) Sketch the wave pattern of the pipe.
(ii) Determine the frequency of the note produced.
(iii) Given that the end correction of the pipe is 0.002m, find the radius of the pipe.
(Take speed of sound in air, c=340ms$^{-1}$)

8. A circular pipe of length 29cm is closed at one end. The air in the pipe resonates when a tuning fork of frequency 860Hz is sounded and held just above the open end. Find the;
(i) Mode of vibration of the pipe.
(ii) End correction of the pipe.
(Take speed of sound in air, c=340ms$^{-1}$)

9.A tube of length 1m has its lowest resonance frequency at 86.2 Hz. With a tube of identical dimensions but open at both ends, the first resonance position occurs at 171Hz. Calculate the; (i) Speed of sound. [v=347.646ms-1]
(ii) End correction. [c=0.00825m]

## BEATS

When two notes with slightly different frequencies but equal amplitude are sounded together, they interfere with one another, and the resultant effect in the sound where sound increases and decreases periodically i.e maximum sound alternating with minimum sound is heard. This phenomenon is known as beats.
**Beats are periodic rise and fall in the intensity of sound.**

The frequency of beats (Beat frequency) is the number of intense sounds heard per second.
The variations in amplitude (and intensity) are called beats. The number of times the sound reaches maximum intensity per second is called the beat intensity.

The production beats is a wave effect explained by the principle of super position. Beats are due to interference in time because the sources are not coherent (they are of different frequency), there is sometimes *reinforcement at* a given time and at other times *cancellation* (amplitude is zero),
Suppose the beat period (i.e time between 2 successive maximum) is T, and that one wave train of frequency $f_1$, makes one cycle more than that of frequency $f_2$.

- Number of cycles of frequency, $f_1$, $=f_1T$
- And number of cycles of frequency $f_2 = f_2T$

$f_1T - f_2T = 1$
$(f_1 - f_2)T = 1$
$$f_1 - f_2 = \frac{1}{T}$$
But, $\frac{1}{T} = f_{bBeat}$ = Beat frequency.
$|\boldsymbol{f_1 - f_2}| = \boldsymbol{f_{bBeat}}$
$\Leftrightarrow (\boldsymbol{f_1 - f_2}) = \boldsymbol{f_{bBeat}}$ ***or*** $\Leftrightarrow -(\boldsymbol{f_1 - f_2}) = \boldsymbol{f_{bBeat}}$

**Note:**
If the beat frequency **increases** after loading $f_1$, then $\boldsymbol{f_2 > f_1}$.
If the beat frequency **decreases** after loading $f_1$, then $\boldsymbol{f_2 < f_1}$.

***Uses of beats.***
1. Beats are used to tune an instrument to a given note. As the instrument note approaches a given note, beats are heard. The instrument may be regarded as tuned when beats occur at a very slow rate.

2. To measure frequency $f_1$ of a given note. A note of known frequency $f_2$ is used to provide beats with the unknown note and the frequency $f$ is obtained by counting the number beats made in a given time. Hence $|f_1 - f_2| = f$. To decide which value of $f_1$ is correct, the end of the unknown tuning fork prong is loaded with a small piece of plasticine which diminishes the frequency a little and the two notes are sounded together again. If the beat frequency increases, then $f_2$ is greater than $f_1$. If it decreases, then, $f_1 \rangle f_2$.
**Note:*Loading the tuning fork slightly lowers its frequency*.**

**Examples**
1. A tuning fork of unknown frequency and a standard fork of 440Hz are sounded simultaneously and beats of frequency 4Hz are heard.
(i) What deduction can you make regarding the frequency of the unknown fork?

**Solution**
Let the un known frequency be $\boldsymbol{f_1}$
$|f_1 - f_2| = f$
$|f_1 - 440| = 4$
Either;
$+(f_1 - 440) = 4 \Leftrightarrow f_1 = 444\text{Hz}$
Or:
$-(f_1 - 440) = 4 \Leftrightarrow f_1 = 436\text{Hz}$

(ii) A small piece of wax is attached to the prongs of the unknown fork and the two forks are sounded again. It is found that the beat frequency is now 3Hz. What deduction can you make, explain your reason.
*Since the beat frequency decreases from 4Hz to 3Hz, this implies that* $f_1 \rangle f_2$
*Hence* $f_1 = 444\,Hz$

2. Beats are produced by a plucked stretched wire and a resonance tube closed at one end each sounding at its fundamental note. The air column has a length of 0.168m the end correction being 0.012m. The wire has a vibrating length of 0.27m and is under tension of 100N. The mass of the position of the wire is $4\text{x}10^{-4}$kg.
(i) Calculate the frequency of the beats heard. If velocity of sound in air columnis350ms$^{-1}$.
For the closed tube,
$$\frac{ë_1}{4} = l + c \Leftrightarrow ë_1 = 4(l + c) = 4(0.168 + 0.012) = 0.72\text{ m}$$
$$f_1 = \frac{v}{ë_1} = \frac{350}{0.72} = 486.1\text{Hz}$$

For the string
$$f_1' = \frac{1}{2l}\sqrt{\frac{T}{ì}} = \frac{1}{2(0.27)} \times \sqrt{\frac{100}{\left(\frac{4\times10^{-2}}{0.27}\right)}} = 481.1\text{Hz}$$
Hence, Beat frequency $= f_1 - f_1'$
$\Leftrightarrow f_{Beat} = (486.1 - 481.1) = 5\text{Hz}$

(ii) Calculate the change in tension of the wire that would make the frequencies of the two notes the same.
Using f = 486.1Hz
For; $f_1 = f_1'$

$$486.1 = \frac{1}{2l}\sqrt{\frac{T}{\text{ì}}}$$

$$486.1 = \frac{1}{2(0.27)} \times \sqrt{\frac{T}{\left(\frac{4\times10^{-2}}{0.27}\right)}}$$

$$T = 102\ N$$

$\underline{\text{Hence, the change in tension } 102 - 100 = 2\text{N}}$

***Questions***

1. Two tuning forks X and Y are sounded together to produce beats of frequency 8Hz. Fork X has a known frequency of 512Hz. When Y is loaded with a small plasticize beats at frequency of 2Hz are heard when the two tuning forks are sounded together.
Calculate the frequency of Y when unloaded.(520Hz)

2. When determining the frequency of a note produced by a guitar, the guitar is sounded together with a tuning fork of frequency 612Hz and 10 beats are heard after 5 seconds. When the tuning fork is loaded with placiticine and the guitar sounded with the fork again, the same number of beats is heard in 2 seconds.
Determine the frequency of the note produced by the guitar. ($f_g$= 614Hz)

3. The wire of a sonometer of mass per unit length $10^{-3}$kgm$^{-1}$ is stretched on the two bridges by a load of 40N. When the wires is struck at the center point so that it executes its fundamental vibration, and at the same time a tuning fork of 264Hz is sounded and beats are heard and found to have a frequency of 3Hz. If the load is slightly increased, the beat frequency is lowered. Calculate the separation of the standing wave. ($l$= 0.38m).

4. Two open pipes of length 50cm and 51cm respectively are sounding their fundamental notes together. Neglecting the end correction, determine the frequency of the beats produced.
(Take speed of sound in air, c=340ms$^{-1}$)

## DOPPLER'S EFFECT.

The pitch of the note from the siren (whistle) of a fast traveling ambulance or train appears to a stationary observer to drop suddenly as it recedes.

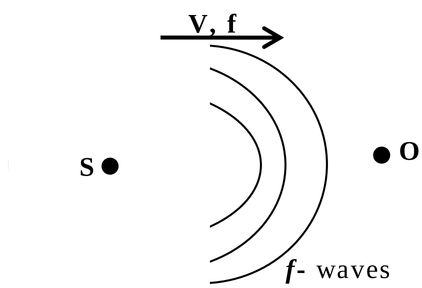
***Doppler Effect*** is the apparent change in the frequency of a wave motion when there is relative motion between the source and observer**.**
It occurs with electromagnetic waves and sound waves.

**Doppler Effect in sound waves.**
**Calculation of apparent wavelength, ë$^1$and apparent frequency, f$^1$.**
The following symbols will be used i.e,
V = speed of sound in air
$u_s$ =speed of source, S
$u_o$ =speed of observer, O
f=frequency of sound waves.
f$^1$= apparent frequency,

Consider a wave of frequency f and velocity, V moving with $u_s$from the source. Let $u_o$ be the velocity of the observer.
Then, note:

- For V and $u_s$ or $u_o$ moving in the <u>same direction</u>, then; $u_s = (+u_s)$and $u_o = (+u_o)$.
- For V and $u_s$ or $u_o$ moving in <u>opposite directions</u>, then; $u_s = (-u_s)$and $u_o = (-u_o)$.
- Velocity of the wave relative to the source is equal to the distance occupied by f- waves.
  $_w\mathbf{V_s} = \boldsymbol{V} - (\pm \boldsymbol{u_s})$.
- Velocity of the wave relative to the observer is equal to the distance occupied by f- waves sent out by the source..
  $_w\mathbf{V_o} = \boldsymbol{V} - (\pm \boldsymbol{u_o})$.
- Apparent wave length of the wave λ$^1$ is given by;
  $_w\mathbf{V_s} = \boldsymbol{f}\text{ë}^{\mathbf{1}} \Leftrightarrow \text{ë}^{\mathbf{1}} = \frac{\boldsymbol{wVs}}{\boldsymbol{f}}$
- Apparent frequency $\mathbf{f^1}$of the wave is given by;
  $_w\mathbf{V_o} = \mathbf{f^1}\text{ë}^{\mathbf{1}} \Leftrightarrow \mathbf{f^1} = \frac{\boldsymbol{wVo}}{\text{ë}^{\mathbf{1}}}$
  $\Leftrightarrow \mathbf{f^1} = \left(\frac{\boldsymbol{wVo}}{\boldsymbol{wVs}}\right)\boldsymbol{f}$

***(a) <u>Stationary source and observer.</u>***



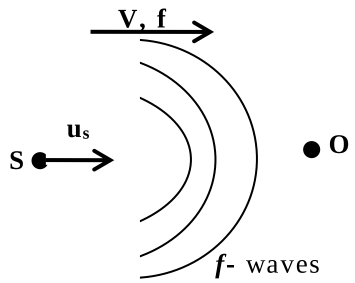

*f*-Waves emitted per second occupy a distance, V and hence the wave length $\lambda^1 = \frac{V}{f}$

Apparent frequency, $\mathbf{f^1} = \left(\frac{\boldsymbol{wVo}}{\boldsymbol{wVs}}\right)\boldsymbol{f}$

Apparent frequency, $\text{f}^1 = \left(\frac{V - (\pm u_o).}{V - (\pm u_s).}\right)f$

Apparent frequency, $\text{f}^1 = \left(\frac{V - 0.}{V - 0.}\right)f = \boldsymbol{f}$

***b)<u>Source moving towardsastationary observer.</u>***




$f$ -Waves are compressed into a smaller distance; $\boldsymbol{V-(+u_s)}$.per second .

The apparent wave length $\lambda' = \dfrac{V-u_s}{f}$

Apparent frequency, $f_1 = \underline{\text{velocity of waves relative to O.}}$
Apparent wave length, λ'.

Apparent frequency, $\mathbf{f^1} = \left(\dfrac{\boldsymbol{wVo}}{\boldsymbol{wVs}}\right)\boldsymbol{f}$

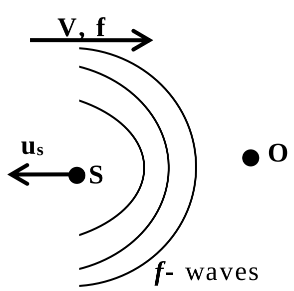
Apparent frequency, $\mathrm{f}^1 = \left(\dfrac{V-(+u_o).}{V-(+u_s).}\right)f$

Apparent frequency, $\mathrm{f}^1 = \left(\dfrac{V-(0).}{V-u_s.}\right)f$

Therefore, $f_1 = \dfrac{V}{\left(\dfrac{V-u_s}{f}\right)} = \dfrac{Vf}{V-u_s}$ , hence $f_1 \rangle f$.

If the source is moving with uniform speed, the pitch of the note is constant but *higher* than the true pitch.

**c) *<u>Source moving away from a stationary observer</u>***




$f$-Waves are compressed into a smaller distance; $\boldsymbol{V-(-u_s)}$per second.

Apparent wave length, $\lambda_2 = \dfrac{V+u_s}{f}$

Apparent frequency, $\mathbf{f^1} = \left(\dfrac{\boldsymbol{wVo}}{\boldsymbol{wVs}}\right)\boldsymbol{f}$

Apparent frequency, $\mathrm{f}^1 = \left(\dfrac{V-0.}{V-(-u_s).}\right)f$

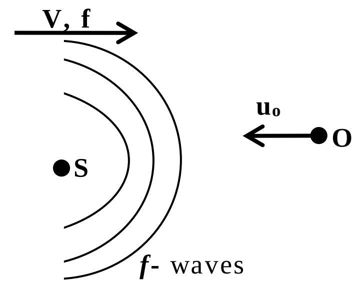
Apparent frequency, $\mathrm{f}^1 = \left(\dfrac{V-(0).}{V+u_s.}\right)f = \left(\dfrac{V.}{V+u_s.}\right)f$

$f^1 < f$ (i.e. observer hears a note of lower pitch than the true pitch).

Apparent change in frequency heard by the observer as the source passes is;

$$f_1 - f_2 = \frac{Vf}{V-u_s} - \frac{Vf}{V+u_s} = \frac{2Vu_s f}{V^2 - u_s^2}$$

**d) *<u>Observer moving towards a stationary source.</u>***




The velocity of the wave relative to O is V+$u_o$. Apparent
$f$-Waves are compressed into a smaller distance;
$\boldsymbol{V-(-u_s)=V}$ Per secondsince the source is stationary.

Apparent wave length, $\lambda_2 = \dfrac{V+u_s}{f}$

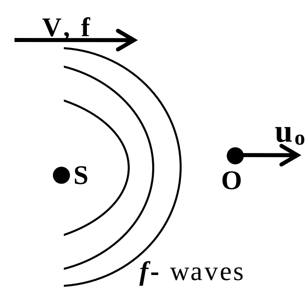
Apparent frequency, $\mathbf{f^1} = \left(\dfrac{\boldsymbol{wVo}}{\boldsymbol{wVs}}\right)\boldsymbol{f}$

Apparent frequency, $\mathrm{f}^1 = \left(\dfrac{V-(-u_o).}{V-(-u_s).}\right)f$

Apparent frequency, $\mathrm{f}^1 = \left(\dfrac{V+u_o.}{V-0.}\right)f = \left(\dfrac{\boldsymbol{V+u_o.}}{\boldsymbol{V.}}\right)\boldsymbol{f}$

$\boldsymbol{f_1 < f}$.(Implying that pitch is higher than the true pitch).

**e*<u>) Observer moving away from a stationary source.</u>***



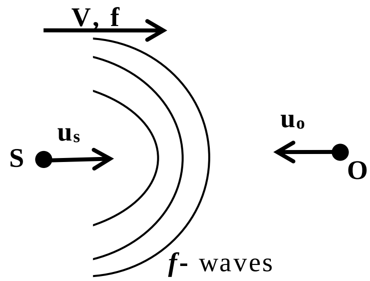

Apparent frequency, $\mathbf{f^1} = \left(\dfrac{\boldsymbol{wVo}}{\boldsymbol{wVs}}\right)\boldsymbol{f}$

Apparent frequency, $\mathrm{f}^1 = \left(\dfrac{V-(+u_o).}{V-(0).}\right)f$

Apparent frequency, $\mathrm{f}^1 = \left(\dfrac{V-u_o.}{V-0.}\right)f = \left(\dfrac{\boldsymbol{V+u_o.}}{\boldsymbol{V}}\right)\boldsymbol{f}$

$f^1 < f$ ;

***f)*** ***Source and Observer moving towards each other.***




$f$-Waves are compressed into a smaller distance;

$$\boldsymbol{V-(-u_s)=V}$$

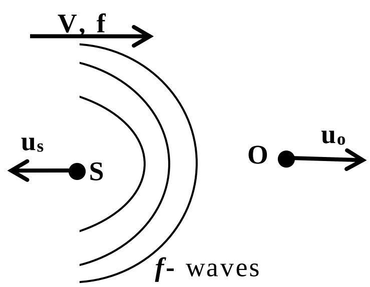
The wave length of the waves reaching O is $\lambda' = \dfrac{V-u_s}{f}$ .

Apparent frequency, $\mathbf{f^1} = \left(\dfrac{\boldsymbol{wVo}}{\boldsymbol{wVs}}\right)\boldsymbol{f}$

Apparent frequency, $\mathrm{f}^1 = \left(\dfrac{V-(-u_o).}{V-(+u_s).}\right)f$

Apparent frequency, $\mathrm{f}^1 = \left(\dfrac{V+u_o.}{V-u_s.}\right)f = \left(\dfrac{\boldsymbol{V+u_o.}}{\boldsymbol{V-u_s.}}\right)\boldsymbol{f}$

***g)*** ***Source and observer moving away from each other.***



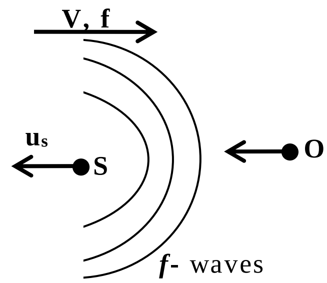

Velocity of waves relative to O =V-$u_o$

Apparent frequency, $\mathbf{f^1} = \left(\dfrac{\boldsymbol{wVo}}{\boldsymbol{wVs}}\right)\boldsymbol{f}$

Apparent frequency, $\mathrm{f}^1 = \left(\dfrac{V-(+u_o).}{V-(-u_s).}\right)f$

Apparent frequency, $\mathrm{f}^1 = \left(\dfrac{V-u_o.}{V+u_s.}\right)f = \left(\dfrac{\boldsymbol{V-u_o.}}{\boldsymbol{V+u_s.}}\right)\boldsymbol{f}$

***h)*** ***Observer moving towards a source moving away from the observer.***




Apparent frequency, $\mathbf{f^1} = \left(\dfrac{\boldsymbol{wVo}}{\boldsymbol{wVs}}\right)\boldsymbol{f}$

Apparent frequency, $\mathrm{f}^1 = \left(\dfrac{V-(-5._o).}{V-(-u_s).}\right)f$

Apparent frequency, $\mathrm{f}^1 = \left(\dfrac{V+u_o.}{V+u_s.}\right)f = \left(\dfrac{\boldsymbol{V+u_o.}}{\boldsymbol{V+u_s.}}\right)\boldsymbol{f}$

In general, apparent frequency $f_1$ is given by

$$f^1 = \frac{V-u_o}{\lambda'} = \frac{(V \pm u_o)\,f}{V \mp u_s}$$

Upper sign applies to approach.
Lower sign applies to moving away from each other (Receding).

**Application of Doppler Effect in Light.**

a) *Rader speed trap*

(*Measurement of speed of a moving car*)

It can be used to determine the speed of a moving car by measuring the shift in the frequency of the micro-waves reflected by the car.

Microwaves of frequency f from stationary radar set are directed to a car moving with speed $u_0$.

Micro waves reflected from the car interfere with those moving from the radar set and they form beats of beat frequency $\Delta f$. where$\Delta f$ is Dropper shift in frequency.

$$\frac{Df}{f} = \frac{2u_0}{C}$$

b) *Reflection of Double stars.*

A star is a sourceof light observed from the earth.

A double star is two stars which are close together that they appear as a single star- even when viewed using large telescopes.

If one star is approaching the earth and the other moving away, from the spectrum of light reflected from what appears as a single star appears between the RED and BLUE SHIFTS. This reveals that they are actually two stars.

c) *Measurement of the speed of a star*

A photograph of a star is taken and the wavelength of light ë$^1$ emitted by the star determined from the spectrum, lines.

A spark spectrum element known to be present in the star is also photographed in the laboratory on earth and the corresponding wavelength measured, the shift $\Delta ë = ë^1 - ë$ measured.
The speed of the star is obtained from $\frac{\Delta ë}{ë} = \frac{U_S}{C}$

NOTE; If$\Delta$ë is negative, the star moving towards the earth. Hence ë$^1$ is displaced (shifted) towards the blue end.

d) *Measurement of plasma temperatures*
Very high temperatures of gases involving fusion reactions can be measured by measuring the broadening or shifting in the spectral lines of the spectra emitted by these gases

Then, $^{dë}/_{ë} = {}^{2u_s}/_{C}$ … … … … … … ….(i)

Also$\frac{1}{2}mu^2 = \frac{3}{4}RT$

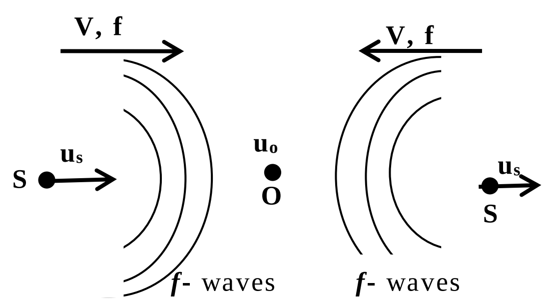
$T = \frac{mu_S^2}{3R}$………………..... (ii)
From (i) and (ii)
$$T = \frac{m}{3R}\left(\frac{C\Delta ë}{2ë}\right)^2 \Leftrightarrow \boldsymbol{T = \frac{mC^2}{12R}\left(\frac{\Delta ë}{ë}\right)^2}$$

NOTE:
- A star is a source of light observed from the earth
- If a star moves away from the earth, the frequency of light decreases as observed from the earth due to Doppler effect
- This apparent decrease in frequency shifts the frequency towards the red end of the visible spectrum (REDSHIFT)

If the star moves towards the earth, frequency increases and shifts towards the blue end of the spectrum (BLUESHIFT)

**Examples**
1. A stationary observer notices the pitch of a police car changes in the ratio of 4:3 when passing him. If the speed of sound is 350ms$^{-1}$, calculate the speed of the car.

**Solution**




$$f_1 = \left(\frac{wVo}{wVs}\right)f \Leftrightarrow f_1 = \frac{(350-0)f}{350-u_s} = \frac{350f}{(350-u_s)} \dots..(i)$$

$$f_2 = \left(\frac{wVo}{wVs}\right)f \Leftrightarrow f_2 = \frac{(350-0)f}{350-(-u_s)} = \frac{350f}{(350+u_s)} \dots(ii)$$

Equation (i) divided by equation (ii) gives
$$\frac{f_1}{f_2} = \frac{4}{3} \Leftrightarrow \frac{4}{3} = \frac{(350+u_s)}{(350-u_s)}$$
$$4(350-u_s) = 3(350+u_s)$$
$$u_s = 50\ ms^{-1}$$

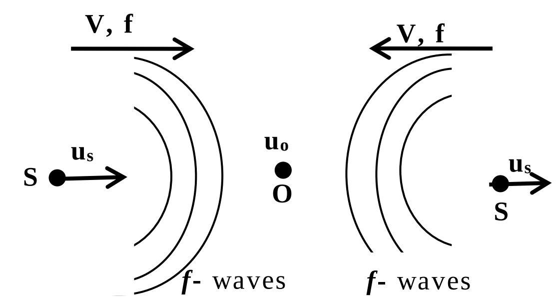
**Alternatively;**
$$f_1 = \left(\frac{wVo}{wVs}\right)f \Leftrightarrow 4 = \frac{(350-0)f}{350-u_s} = \frac{350f}{(350-u_s)}$$
$$350f = 4(350-u_s)$$
$$350f + 4u_s = 1400 \dots\dots\dots\dots\dots\dots..(i)$$

$$f_2 = \left(\frac{wVo}{wVs}\right)f \Leftrightarrow 3 = \frac{(350-0)f}{350-(-u_s)} = \frac{350f}{(350+u_s)}$$
$$350f = 3(350+u_s)$$
$$350f + 3u_s = 1050 \dots\dots\dots\dots\dots\dots..(ii)$$

Solving equations (i) and (ii) simultaneously gives;
$$\boldsymbol{u_s = 50\ ms^{-1}}$$

2. An observer moving between two identical sources of sound along the straight line joining them hears beats at the rate of 4s$^{-1}$. At what velocity is he moving if the frequency of each source is 500Hz and the velocity of sound when he makes the observation is 340ms$^{-1}$.

**Solution**




1$^{st}$source moving towards the observer;
$$f_1 = \left(\frac{wVo}{wVs}\right)f \Leftrightarrow f_1 = \frac{[350-(-u_s)]f}{340-0} = \frac{(340+u_0)\times 500}{340}$$

2$^{nd}$ source moving away from the observer;
$$f_2 = \left(\frac{wVo}{wVs}\right)f \Leftrightarrow f_2 = \frac{(350-u_0)f}{350-0} = \frac{(340-u_0)\times 500}{340}$$

$$f_{Beat} = f_1 - f_2$$
$$4 = f_1 - f_2$$

$$4 = \left(\frac{(340 + u_0) \times 500}{340}\right) - \left(\frac{(340 - u_0) \times 500}{340}\right)$$

$$4 = \frac{500}{340}[(340 + u_0) - (340 - u_0)]$$

$$\underline{\underline{u_0 = 1.36\ \mathbf{ms^{-1}}}}$$

***Questions***

1. Two observers A and B are provided with sources of sound of frequency 500Hz. A remains stationary and B moves away from him at a velocity of 1.8ms$^{-1}$. How many beats per second are observed by A and by B, the velocity of sound being 330ms$^{-1}$? (2.73Hz)

2. A car sounds its horn when travelling at 15ms$^{-1}$ between two stationary observers A and B. Observer A hears sound at a frequency of 538Hz, while observer B hears a lower frequency. Calculate the;
 (i) Actual frequency of the sound. ($f$=514.265Hz)
 (ii)Frequency heard by observer B. ($f_B$=492.5Hz)
(Given speed of sound in air = 340ms$^{-1}$)

## ③ WAVE THEORY

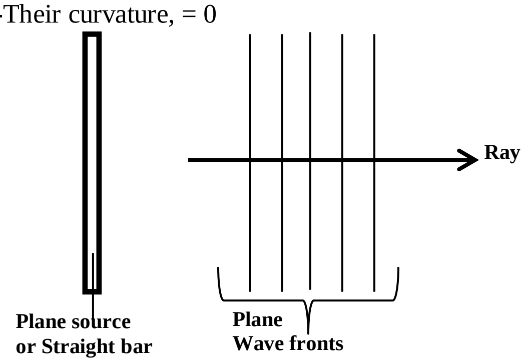
This deals with the behavior of waves, including reflection, refraction, wave fronts, Optical path, and Huygens's principle.

Wave Fronts: A wave front is a surface or section taken through an advancing wave along which all particles are in phase.
A long a wave front, every particle transmitting the wave is at the same distance from the source of the wave and is in the same state of disturbance.

Types of wave fronts

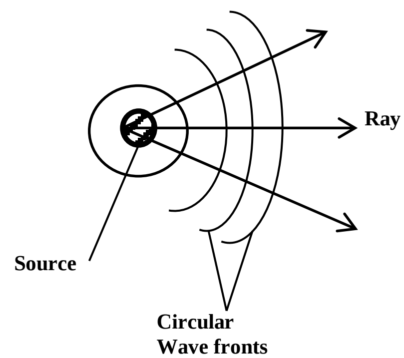
1. Plane wave fronts

Here , the wave fronts are parallel to each other.
They can be obtained when the surface of water is disturbed by a plane object like a ruler.
-They normally come from a distant source.
-Their radius of curvature, $r = \infty$
-Their curvature, = 0




2. Circular wave fronts

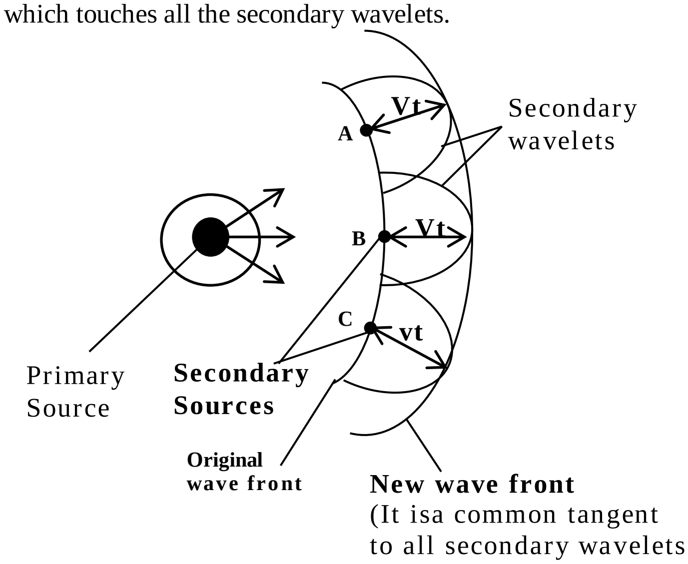
These consist of wave fronts that are concentric circles with the source of the wave at the Centre.
They can be obtained when the surface of water is disturbed by a round or spherical object like a marble.
-They normally come from a nearby source.
-Have radius of curvature positive for a concave surface and negative for a convex surface.
-The curvature is positive for a concave surface and negative for a convex surface.




A ray: A ray is a line perpendicular to the wave front, which shows the direction of travel of the wave.

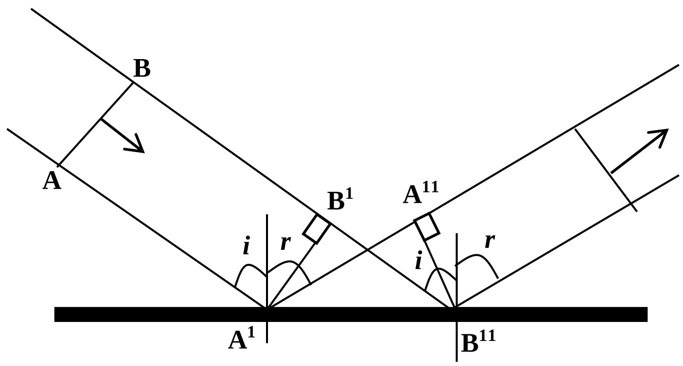
### Huygens's Principle

It states that, every point on a wave front may be a source of secondary spherical (or semi-circular) wavelets which spreads out at the wave speed, and the new wave front is an envelope which touches all the secondary wavelets.



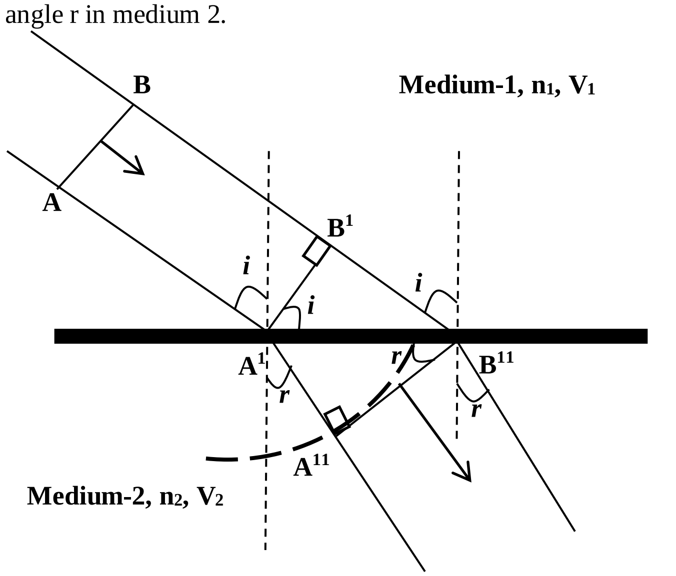

Huygens's Construction applied to reflection:
Consider a plane wave font AB incident obliquely on a reflecting surface XY at an angle i.

- AB forms new centres of disturbance
- After a time t, A will have reached A' while B will have reached B'




From the diagram, triangles $A^1B^1B^{11}$ and $A^1A^{11}B^{11}$ are similar.
Thus; Angle $A^{11}A^1B^{11}$= Angle $A^{11}B^{11}A^1$
$(90 - r) = (90 - i)$
<u>$r = i$</u>
This is one of the laws of reflection.

<u>Huygen's Construction applied to refraction:</u>
Consider a plane wave font AB incident on a refracting surface at an angle i in medium 1 and refracted through an angle r in medium 2.




As the wave is refracted, the frequency of of the wave remains constant.

Refractive index of a medium, n is given by;

$$n_{mediun} = \frac{\text{wave length of a wave in air}}{\text{Wave length of a wave in the medium}}$$

$$\mathrm{n_{mediun}} = \frac{\lambda_{\mathrm{air}}}{\lambda_{\mathrm{medium}}}$$

$$\mathrm{n_{mediun}} = \frac{c/_{f}}{v/_{f}}$$

$$\boldsymbol{n_{mediun}} = \frac{\boldsymbol{c}}{\boldsymbol{v}}$$

Where c = speed of the wave in air or vacuum.
V = speed of the wave in the medium.
n = refractive index of the medium.

- ❖ If light from side $B^1$ reaches the surface at $B^{11}$ in a time t, then:
  **$B^1B^{11} = V_1t$................................(i)**
- ❖ In the same time, t, $A^1$ forms a semi-circular wavelet of radius $A^1A^{11}$ in the second medium.
  **$A^1A^{11} = V_2t$................................(ii)**
- ❖ The new refracted wave front is a tangent to thewavelet at $A^{11}$and the refracted rays are perpendicular to the wave front $A^{11}B^{11}$.

$From\ triangle\ A^1B^1B^{11};\ \sin i = \frac{B^1B^{11}}{A^1B^{11}}$...............(iii)

$From\ triangle\ A^1B^{11}A^{11};\ \sin r = \frac{B^1B^{11}}{A^1B^{11}}$............... (iv)

From equations (i), (ii), (iii) and (iv)

$$\frac{\mathbf{\sin i}}{\mathbf{\sin r}} = \frac{\mathbf{B^1B^{11}}}{\mathbf{A^1A^{11}}} = \frac{\mathbf{V_1 t}}{\mathbf{V_2 t}},\ \text{Where, } \mathbf{V_1} = \frac{\mathbf{c}}{\mathbf{n_1}}\ \mathbf{and}\ \mathbf{V_2} = \frac{\mathbf{c}}{\mathbf{n_2}},$$

Since $V_1$, $V_2$, $n_1$, and $n_2$ are constants for a given pair of media and particular wavelength, it implies that;
$\boldsymbol{n_1 \sin i = n_2 \sin r}$**.** This is Snell's law.

## (A) <u>INTERFERENCE OF LIGHT WAVES.</u>

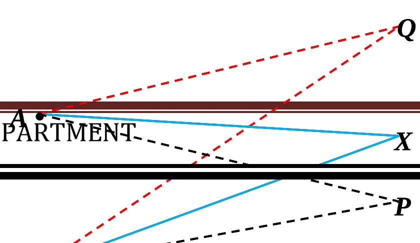
Interference is the superposition of two or more similar waves resulting into regions of maximum and minimum intensity.
The waves must be similar i.e of equal wave length and amplitude.
Interference always occurs whenever two waves come together. However, for interference effects to be observed, certain conditions must be full filled.

**Conditions necessary for interference effects to be observed.**

(i) The sources producing interference must be coherent sources.
(ii) The two coherent sources must be narrow, parallel and close to each other.
(iii) The waves interfering must have approximately the same amplitude.

**Note:**
Coherent sources are sources which emit waves of the same frequency and are in phase (i.e. have zero phase difference).
This is archived by using monochromatic light.

*<u>Conditions for observable interference</u>*



Suppose two light sources A and B have exactly the same frequency and amplitude of vibration and their vibrations are always in phase with each other. Such sources are called coherent sources.
Suppose X is equidistant from A and B, the vibration at X due to the two sources will always be in phase. The distance AX traveled by the wave originating at A is equal to the distance BX traveled by wave originating at B.
Assume that waves from A and B are moving at X have the same amplitude, then the amplitude of the resultant wave at X is twice that of either waves from A and B.
The light energy at X is proportional to the square of the amplitude of resultant wave and hence is four times that due to A or B alone. A bright beam of light is therefore obtained at X as A and B are coherent sources. It is due to constructive interference of light waves at A and B at X i.e a crest from A reaches at the same time as a crest from B.

Suppose $Q$ is a point such that B$Q \rangle AQ$ by a whole number of wave lengths. The waves are moving at $Q$ from A will be in phase as a wave moving at the same point from B, a bright band will be obtained at $Q$.
Therefore condition for constructive interference to occur at any point Y is such that the path difference BY –AY =m$\lambda$ where $\lambda$ is light wavelength from sources A and B and m is an integer.
Consider a point P where distance from B is $1\!/\!_2$ a wave length longer than its distance from A .i.e;

AP –BP = $\dfrac{\lambda}{2}$ where $\lambda$ is a wave length of light from A and B. Then the waves arriving at P from A will be out of phase with waves arriving at P from B. If the waves have equal amplitudes, we obtain the following:-

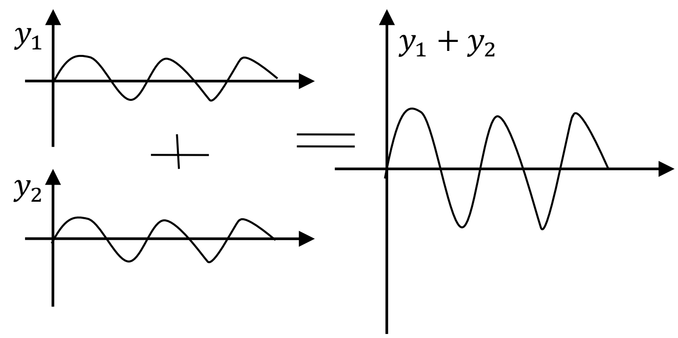
The resultant at P is zero as the displacements at any instant are equal and opposite of each other. No light is therefore seen at P. A dark band is obtained. Destructive interference is said to have occurred.
In general, if P is such that the path difference AP – BP = $\left(m + 1\!/\!_2\right)\lambda$ where m is an integer, destructive interference is said to have occurred at P.

***Coherent sources are those which emit light waves of the same wave length or frequency which are always in phase with each other or have a constant phase.***

**Types of interference**

**(i) Constructive Interference.**

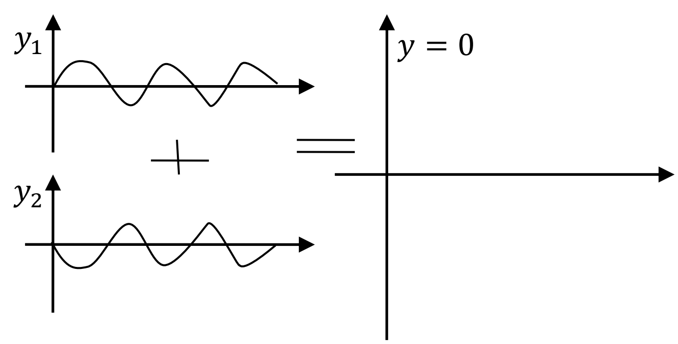
This is the reinforcement of the intensities of two coherent sources to give maximum intensity when the wave disturbances from the two sources are superposed.
This occurs when a crest from one set reinforces a crest from another set or a trough from one source reinforces a trough from another source.




Note: For constructive interference, (Bright band or fringe) to occur, the path difference between the waves should be an integral multiple of the wave length of the light.
i.e. **Path difference = n λ .where n = 0, 1, 2, 3, ……..**
Thus n=0 gives 1st bright fringe, n=1 gives the 2nd bright fringe, e.t.c.

**(ii) Destructive Interference.**

This is the cancellation of the intensities of two coherent waves to give minimum intensity when the wave disturbances from the two wave sources are superposed.
This occurs when a crest from one wave falls on a trough of another wave.




Note: For destructive interference, (Dark band or fringe) to occur, the path difference between the waves should be an odd number multipleofhalf the wave length of the light.
i.e. Path difference = n λ + $\frac{1}{2}$ë
**Path difference = (n + $\frac{1}{2}$)ëWhere n = 0, 1, 2, 3…**
Thus n=0 gives 1stdark fringe, n=1 gives the 2nd bright fringe, e.t.c.

**Optical path.**

Suppose light travels a distance $x$ in a medium of refractive index, n. If λ is the wave length of light in the medium, the quantity $\emptyset = \frac{2\eth x}{ë}$ is the phase difference due to the path.

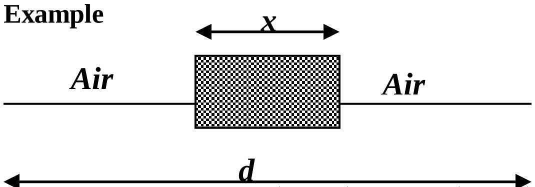
If the speed of light in the median is V then the refractive index of the median, $\boldsymbol{n}=\frac{c}{\lambda}$ where C = speed of light in vacuum. V =speed of light in medium. But c = $\lambda_o f$ , since V =λf, where $\lambda o$ and $\lambda$ are wave length in a vacuum and medium respectively.

$$n=\frac{\lambda_o f}{\lambda f}=\frac{\lambda_o}{\lambda},\ \lambda=\frac{\lambda_o}{n}$$

$$\emptyset=\frac{\mathbf{2}\pi\boldsymbol{x}}{\left(\frac{\lambda_0}{n}\right)}=\frac{\mathbf{2}\pi\boldsymbol{nx}}{\lambda_0}=\left(\frac{\mathbf{2}\pi}{\lambda_0}\right)\boldsymbol{nx}$$

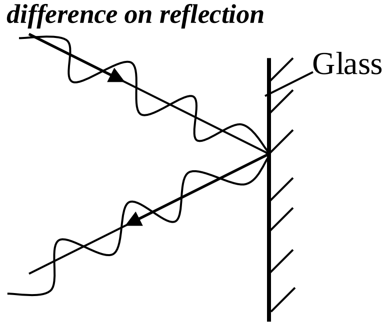
$\boldsymbol{nx}$is called the optical path. It is the product of the refractive index and the length light covers in the medium.

**Example**




Optical path $=nx+1\times(d-x)=nx+(d-x)$

***Phase difference on reflection***

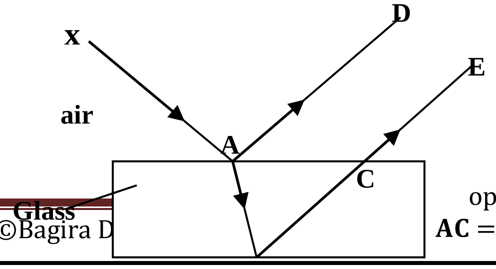



Light waves may undergo a phase change by reflection at some point in their path. If the waves are reflected at a denser medium e.g at the air –glass interface, a phase change of $\pi$ radius compared to the incident waves occurs. This corresponds to a pathdifferenceof$\frac{\lambda}{2}$. **i.e;**

$$\phi=\left(\frac{2\pi x}{\lambda}\right)\Leftrightarrow\pi=\frac{2\pi x}{\lambda}\ \Leftrightarrow\boldsymbol{x}=\frac{\lambda}{\mathbf{2}}$$

- ❖ Reflection occurs at the interface with a denser medium at A. It suffers a phase change of πradians or a path difference of$\frac{\lambda}{2}$.
- ❖ No phase change at B and C since reflection occurs ataless dense medium.

Consider monochromatic light incident on a glass plate as shown.




optical Path
$\mathbf{AC}=\mathbf{n}(\mathbf{AB}+\mathbf{BC})$

The optical path A to C is n (AB+BC).
There is no phase change at B and C since reflection occurs at an interface with a less dense medium but there is a phase change of $\pi$ radians equivalent to a path of $\lambda/2$. When XA is reflected along AD. Therefore the optical path difference between light reflected at A and that reflected at B is;

n (AB+ BC) $=\frac{\lambda}{2}$.

For Reflection at A (i.e. at a denser medium)

❖ For Constructive Interference (bright fringe);

**Path difference** $+\frac{\lambda}{2}=\mathbf{n}\lambda$.Where n =1, 2, 3, ….……..…..

Thus n=0 is impossible. Thus 1st fringe is dark, n=1 gives the 1st bright fringe, e.t.c.

❖ For Destructive Interference(Dark fringe);

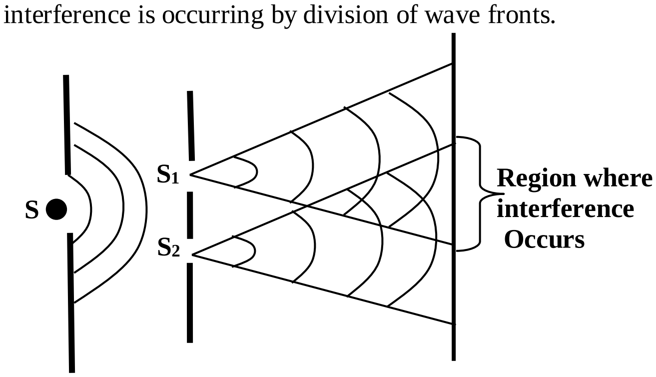
**Path difference** $+\frac{\lambda}{2}=(\mathbf{n}\ +\ \frac{1}{2})\lambda$ **.** Where n = 0, 1, 2, 3,….

**HOW TO PRODUCE TWO COHERENT SOURCES.**

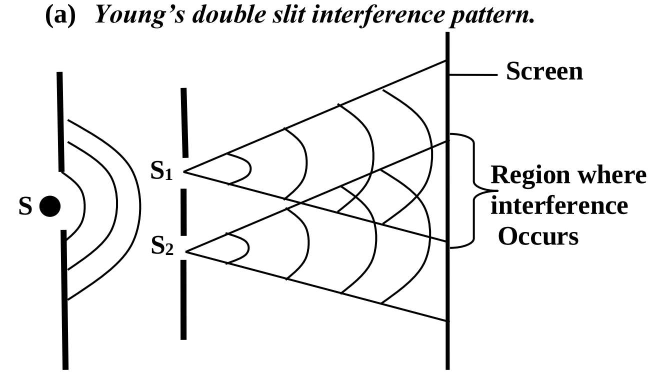
(i) **BY DIVISION OF WAVE FRONT**

e.g Young's double slit interference.

The wave front from S is divided at slits $s_1$ and $s_2$. Hence interference is occurring by division of wave fronts.

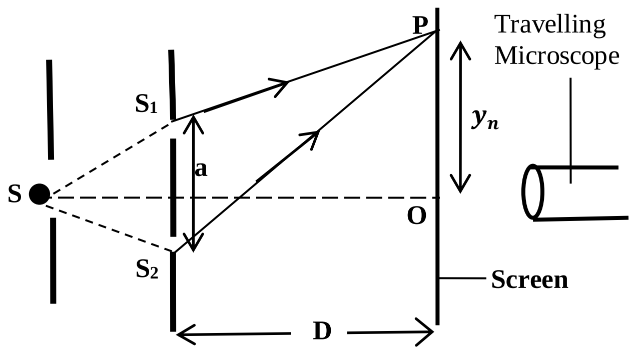



**(a)** ***Young's double slit interference pattern.***




Monochromatic light from a narrow vertical slit S falls on two other narrow slit $S_1$ and $S_2$. Which are very close together and parallel to S. $S_1$ and $S_2$ act as two coherent sources? Diffraction also takes place at $S_1$ and $S_2$. And interference occurs in the region where light from $S_1$ overlaps that from $S_2$. A series of alternate bright and bark equally spaced vertical bands (or fringes) are observed on the screen.

***Young's double slit method for measuring wavelength and velocity of light and comparison of wave length.***




Light of known frequency, **f** from a sodium lamp, S is made to illuminate two slits $s_1$ and $s_2$ as shown in the diagram above.

Alternate regions of bright and dark fringes are observed on the screen using a travelling microscope.

The distances 'a' between the double slits $s_1$ and $s_2$ and 'D'from the double slits to the screen are measured using a travelling microscope and a metre – rule respectively

The fringe separation.$y_n$is obtained by measuring the distance between the fringes and then dividing by the number of fringes observed.

The wave length of light is then calculated from; ë = $\frac{\mathbf{ay}}{\mathbf{D}}$

If f, is the frequency of light, then the speed of light, V is calculated from;**V = f**ë.

**NOTE**

- ♦ To compare wave lengths of different coloured lights, different colour filters are placed between the double slits and the screen.
- ♦ Then, the corresponding wave lengths ë for each colour is measured as described above and then compared.

***Observations made from Young's double slit method for measuring wavelength.***(i. e. from ë = $\frac{\mathbf{ay}}{\mathbf{D}}$).

(i) If the source slit S is moved closer to the double slits $s_1$ and $s_2$, the intensityor brightness of fringes increases but the fringe separation**'y'** is not affected.

(ii) If the source slit S, progressively widened, the fringes gradually disappear.This is because a large slit S equivalent to a large number of narrow slits each producing its own fringe system.The bright and dark bands of the different systems overlap giving rise to uniform illumination

(iii) Increasing the width of any of the slits i.e. S, $s_1$ and $s_2$. This increases the intensity or brightness of the fringes, but they become blurred.

(iv) If any of the double (or twin) slits $s_1$ and $s_2$ is covered, the fringes disappear.

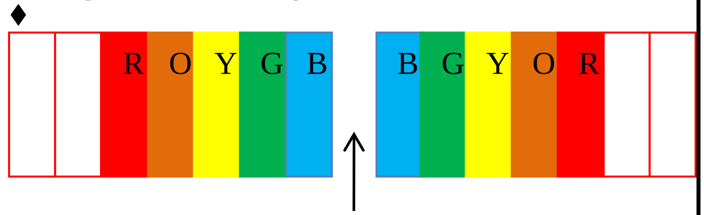
(v) Increasing the distance 'a' between double slits decreases the fringe separation, **y.**$\left(\boldsymbol{i.e.y}\text{á}\frac{\mathbf{1}}{\boldsymbol{a}}\right)$.

(vi) Increasing the distance D of the screen from the double slits increases the fringe separation.($\boldsymbol{i.e.y}$á$\boldsymbol{D}$**).**
But this reduces the intensity of or brightness of the fringe pattern.

(vii) The number of fringes obtained depends on the amount of diffraction occurring at the slits. ($\boldsymbol{a}\,\mathbf{sin}\,$è $= \boldsymbol{n}$ë**).**This also depends on the width of the slits. The narrower the slits, the greater the number of fringes produced due to increased diffraction.($\boldsymbol{a}$ á $\boldsymbol{n}$**).** However, the intensity of the fringes reduces due to the little light that goes through the narrow slits.

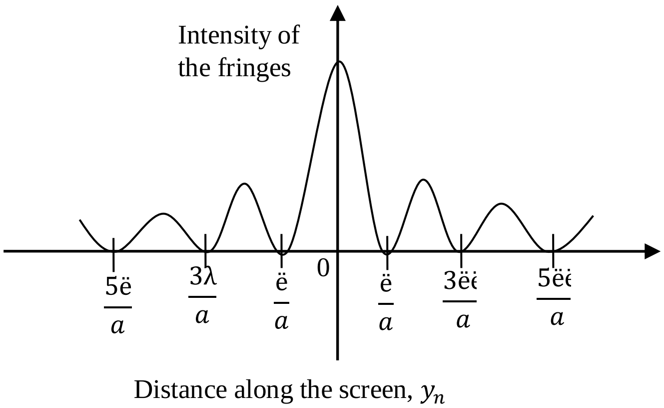
(viii) <u>Effect of using white light instead of monochromatic light.</u>

- ♦ If white light is used, the central fringe is white. This is because; the path difference is zero for all wave lengths of the colours of white light. Therefore each colour produces a bright fringe at the centre 'O'. These colours overlap producing the white central fringe.
- ♦ Further away from the white central band, a pattern of bright coloured fringes is observed with blue fringes appearing first. This is because blue has the shortest visible wavelength.
- ♦ Furthest away from the white central band, overlap of the fringe systems for different orders occur. This leads to an impression of white light.
- ♦




**White central band formed due to overlap of all the colours**

<u>Variation of the fringe intensity with distance along the screen.( For Young's double slit interference.</u>



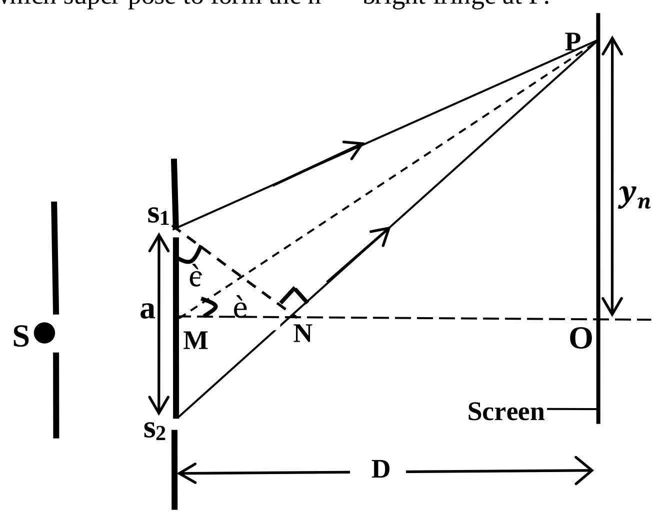

**Qn.What would be the effect of replacing monochromatic light with white light in Young's double experiment?**

Using white light, fewer fringes are seen and each colour produces its own set of fringes which overlap. Only the central fringe is white, its position being the only one where the path difference is zero for all colours. The first coloured fringe is bluish near to the central fringe and red at the far end.

The fringe spacing for red is greater than blue light. Therefore red light must have a greater wave length than blue light since $y\alpha\lambda$ (If **a** and**D** are constant).

**Proof of The fringe spacing,y** $= \frac{\text{ë}\boldsymbol{D}}{\boldsymbol{a}}$

Consider two waves from two coherent sources $s_1$ and $s_2$ which super pose to form the n$^{th}$ – bright fringe at P.




- For the **n$^{th}$** – bright fringe;
  **Path difference,** $\boldsymbol{S_2N} = \boldsymbol{n}$ë ………………….. $(i)$

Triangles$S_1S_2N$ and $MOP$ are similar since $a \ll$D.

♦ From triangle $S_1S_2N$,

$\boldsymbol{S_1S_2 = a}$is very small and **PM** >> **a**. Thus $S_1N$ is nearly perpendicular to $S_2N$. Therefore;

$$\sin \text{è} = \frac{S_2N}{S_1N} = \frac{Path\ difference}{Slit\ seperation, a}$$

$$\sin \text{è} = \frac{S_2N}{a} \Leftrightarrow \quad S_2N = a \sin \text{è}$$

**Path difference,$\boldsymbol{S_2N} = \boldsymbol{a}$ sin è……………………(ii)**

♦ From triangle PMO,

Angles è is very small in radians. Thus sin è ≈ tan è ≈ è

sin è ≈ tan è = $\frac{y_n}{D}$ …………………………………..(iii)

♦ From equations, (i), (ii) and (ii)

$$\frac{n\text{ë}}{a} = \frac{y_n}{D}$$

$$\boldsymbol{y_n} = \frac{\boldsymbol{nD}\text{ë}}{\boldsymbol{a}}$$

Where n = number of bright fringes. n = 1, 2, 3 …

**$y_n$** gives the fringe separation. i.e distance between the nth bright fringe and the centre O. Similarly, $\boldsymbol{y_{(n+1)}}$

Fringe spacing (i.e. distance between two successive fringes) from centre O;

$$y = \boldsymbol{y_{(n+1)}} - \boldsymbol{y_n}$$

$$\text{y} = \frac{(n+1)\text{ë}D}{a} - \frac{n\text{ë}D}{a} -$$

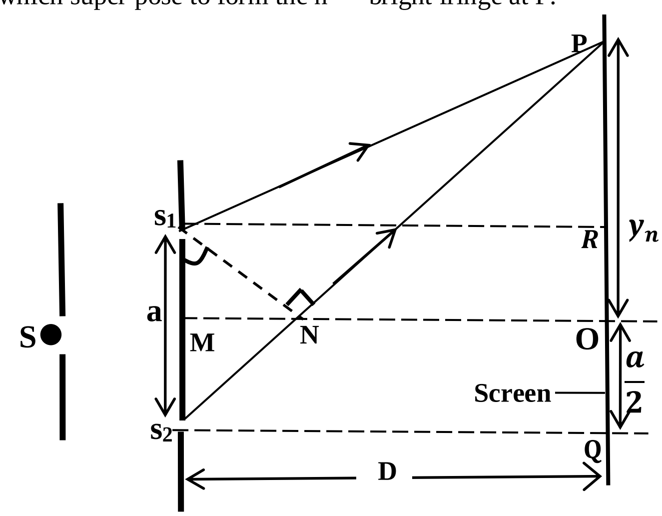
$$\text{y} = \frac{(n+1-n)\text{ë}D}{a}$$

$$\mathbf{y} = \frac{\text{ë}\boldsymbol{D}}{\boldsymbol{a}}$$

- ***The same result above is obtained if a dark fringe is considered.***

<u>Alternative Method</u>

Consider two waves from two coherent sources $s_1$ and $s_2$ which super pose to form the n$^{th}$ – bright fringe at P.




From triangle$S_1RP$

$$\overline{RP}^2 + \overline{S_1R}^2 = \overline{S_1P}^2$$

$$\left(y_n - \frac{a}{2}\right)^2 + D^2 = \overline{S_1P}^2 \ldots\ldots\ldots\ldots\ldots\ldots.(i)$$

From triangle$S_2QP$

$$\overline{QP}^2 + \overline{S_2O}^2 = \overline{S_2P}^2$$

$$\left(y_n + \frac{a}{2}\right)^2 + D^2 = \overline{S_2P}^2 \ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

From equations (i) and (ii)

Path difference = $\overline{S_2P} \quad - \overline{S_1P}$

$$\overline{S_2P}^2 - \overline{S_1P}^2 = \left(y_n + \frac{a}{2}\right)^2 + D^2 - \left[\left(y_n - \frac{a}{2}\right)^2 + D^2\right]$$

$$= 2ay_n \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

$$(\overline{S_2P}^2 - \overline{S_1P}^2)(\overline{S_2P}^2 + \overline{S_1P}^2) = 2ay_n$$

But since, a<<<D, $S_2P \approx S_1P \approx MO = D$

$$(D + D)(S_2P - S_1P) = 2ay_n$$

$$2D(S_2P - S_1P) = 2ay_n$$

$$(S_2P - S_1P) = \frac{ay_n}{D}$$

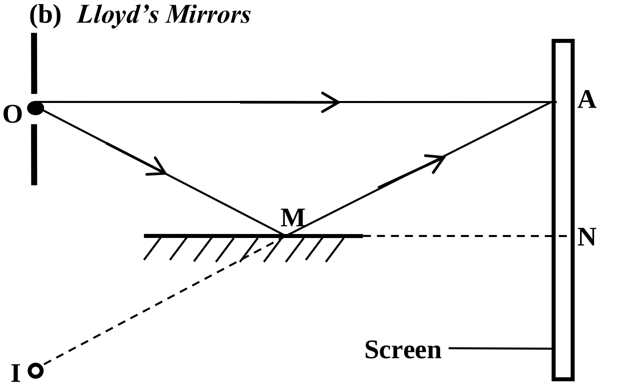
$$(\text{Path difference}, S_2P - S_1P) = \frac{\boldsymbol{ay_n}}{\boldsymbol{D}}$$

For a bright fringe, Path difference**,** $\Delta = \boldsymbol{n}$ë

$$\Leftrightarrow \frac{\boldsymbol{ay_n}}{\boldsymbol{D}} = \boldsymbol{n}\text{ë}$$

$$\boldsymbol{y_n} = \frac{\boldsymbol{n}\text{ë}\boldsymbol{D}}{\boldsymbol{a}}$$

**(b)** ***Lloyd's Mirrors***




The plane mirror M is illuminated by light from slit O, parallel to the mirror.

A point such as, A on the screen is illuminated by waves from, O travelling a long;

(i) OA
(ii) OM and reflected along MA, which appear to come from the virtual image, I of O in the mirror.

Since O and I are close, coherent sources interference fringes are formed on the screen.

Fringes at N (intersection of mirrors and screen) are dark. Since ON = IN, this fringe might have been expected to be bright before the experiment.

However, because of reflection at the mirror surface which is a denser surface than air, a phase change of $180^0$, equivalent to half the wave length occurred.

**(c)** ***Fresnel's Bi-prism Single slit interference pattern.***

This is another example of optical interference by division of wave fronts.

- Monochromatic light from a single slit, S falls on a double glass bi-prism as shown below.
- Two virtual images of the slit, S are formed at $s_1$ and $s_2$ due to refraction at each half of the bi-prism.
- These virtual images act as coherent sources which are close together.
- When the waves from $s_1$ and $s_2$ are superposed, alternate dark and bright fringes are formed on the screen when viewed from a low power microscope.

❖ Because of reflection at a denser medium (glass)

For constructive Interference;

$$\Delta + \frac{\text{ë}}{2} = n\text{ë}$$

$$\Delta = n\text{ë} - \frac{\text{ë}}{2}$$

$$\text{Ä} = \left(\boldsymbol{n} - \frac{\mathbf{1}}{\mathbf{2}}\right)\text{ë}, \boldsymbol{For\ n = 1, 2, 3}, \ldots \ldots$$

For destructive Interference;

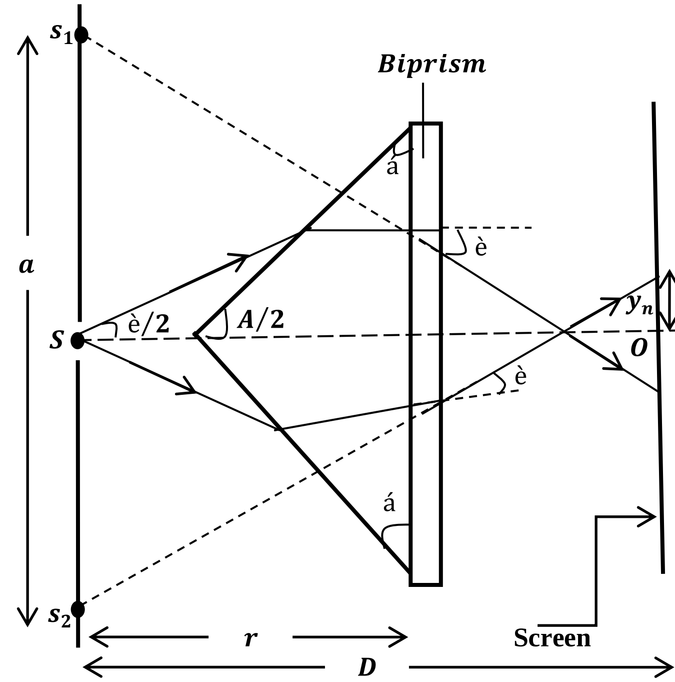
$$\Delta + \frac{\text{ë}}{2} = (n + \frac{1}{2})\text{ë}$$

$$\Delta = \left(n + \frac{1}{2}\right)\text{ë} - \frac{\text{ë}}{2}$$

$$\text{Ä} = \boldsymbol{n}\text{ë}, \quad \boldsymbol{For\ n = 0, 1, 2, 3}, \ldots \ldots$$

❖ At O, the path difference for waves from $s_1$ and $s_2$ is zero. So we would expect a bright fringe at O.

But because of reflection at a denser medium, light suffers a phase change of $\mathbf{180^0}(\boldsymbol{or}\ \text{ð}\ \boldsymbol{radians})$ which is equivalent to a path difference of $\frac{\mathbf{1}}{\mathbf{2}}$ ë**.**Hence a dark band is formed at O instead of a bright band.




From the diagram; $\mathbf{tan}\left(\frac{\text{è}}{\mathbf{2}}\right) = \frac{\boldsymbol{(a/4)}}{\boldsymbol{r}}$

In the diagram, á, è, $and\ A$ are small angles in radians.

Thus; tan è $\approx$ sin è $\approx$ è

$\Leftrightarrow \left(\frac{è}{2}\right) = \frac{(a/4)}{r}$

$\underline{\Leftrightarrow \boldsymbol{a = 2r}è}$

But for thin lenses or prisms the deviation, è is given by;

$è = (n-1)á$

$\underline{\Leftrightarrow a = 2r(n-1)á \qquad \Leftrightarrow \boldsymbol{a = 2r}á\boldsymbol{(n-1)}}$

Whereáis the small refracting angle of the bi-prism in radians.

$2á + A = 180 \Rightarrow á = \frac{\boldsymbol{(180-A)}}{\boldsymbol{2}}$

The path difference, $\Delta = \frac{ay_n}{D}$

The fringe separation, $y_n = \frac{nëD}{a}$

The fringe spacing, $\Delta\mathbf{y} = \frac{ë\boldsymbol{D}}{\boldsymbol{a}}$

The same result is obtained if a dark fringe was used instead.

<u>An advantage of Bi- prism over Young's double slit method in measuring the wave length of light.</u>

❖ Bi-prism produces brighter fringes than those in Young's double slits.
  This is because the prism converges most of the light onto the screen.

**Examples:**

**1.** Two slits 2.5mm apart are placed at a distance of 1m from the screen. The slits are illuminated with light of wave length 550nm. Calculate the;
  (i) Fringe spacing
  (ii) Distance between the 4th and 2nd bright fringe of the interference pattern.

**Solution:**

Given; a = 2.5mm = 2.5 X $10^{-3}$m, D = 1.0m,

$ë = 550nm = 550 \times 10^{-9}m.$

(i) Fringe spacing

$$y = \frac{ëD}{a}$$

$$y = \frac{(550 \times 10^{-9})(1.0)}{2.5 \times 10^{-3}}$$

$$\underline{y = 2.2 \times 10^{-4}m}$$

(ii) Distance between the 4th and 2nd bright fringe.

For the nth - bright fringe, $\boldsymbol{y_n} = \frac{\boldsymbol{n}ë\boldsymbol{D}}{\boldsymbol{a}}$

For the 2nd - bright fringe, $\boldsymbol{y_2} = \frac{\boldsymbol{2}ë\boldsymbol{D}}{\boldsymbol{a}} = \left(\frac{\boldsymbol{2}\times 550\times 10^{-9}\times 1.0}{2.5\times 10^{-3}}\right) =$

For the 4th - bright fringe, $\boldsymbol{y_4} = \frac{\boldsymbol{4}ë\boldsymbol{D}}{\boldsymbol{a}} = \left(\frac{\boldsymbol{4}\times 550\times 10^{-9}\times 1.0}{2.5\times 10^{-3}}\right) =$

Distance between the 4th and 2nd bright fringe.

$${}_4y_2 = y_4 - y_2$$

$${}_4y_2 = \frac{4ëD}{a} - \frac{2ëD}{a}$$

$${}_4y_2 = \frac{(4-1)ëD}{a}$$

$${}_4y_2 = \frac{2ëD}{a} = \left(\frac{\mathbf{2} \times 550 \times 10^{-9} \times 1.0}{2.5 \times 10^{-3}}\right)$$

$${}_4y_2 = \mathbf{4.4 \times 10^{-4}}\boldsymbol{m}$$

**2.** In Young's double slit experiment, the 8th bright fringe is formed 5mm away from the centre of the fringe system when the wave length of light used is$5.2 \times 10^{-7}$m. Calculate the;
  (i) Separation of the two slits if the distance from the slits to the screen is 80cm.
  (ii) Find the total number of fringes observed
  (iii) How far from the centre is the **7th** dark fringe formed.

**Solution:**

Given;$y_2 = 5\text{mm} = 5\text{ X }10^{-3}\text{m}, \text{D} = 80\text{cm} = 80 \times 10^{-2}\text{m}$,$ë = 5.2 \times 10^{-7}$m. .a =?

(i) Slit Separation, a :

From;$y_n = \frac{nëD}{a}$

$$5\text{ X }10^{-3} = \frac{8(5.2 \times 10^{-7})(80 \times 10^{-2})}{a}$$

$$a = \frac{8(5.2 \times 10^{-7})(80 \times 10^{-2})}{5\text{ X }10^{-3}}$$

$$\underline{\underline{a = 7.94 \times 10^{-4}m}}$$

**(ii)** the total number of fringes observed

The total number of fringes observed =2m+1

$2\text{m} + 1 = 2(8) + 8 = 17\ fringes$.i.e. eight fringes on either sides of the central minimum

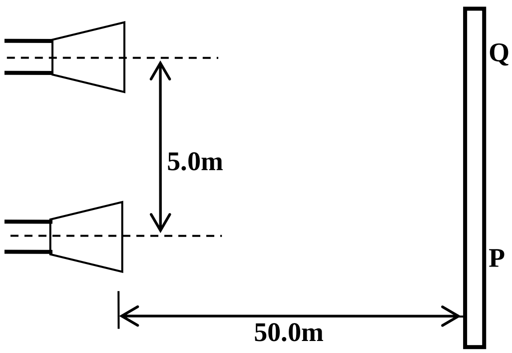
(iii) For a dark fringe formed.

$$y_n = \frac{(n - \frac{1}{2})ëD}{a}$$

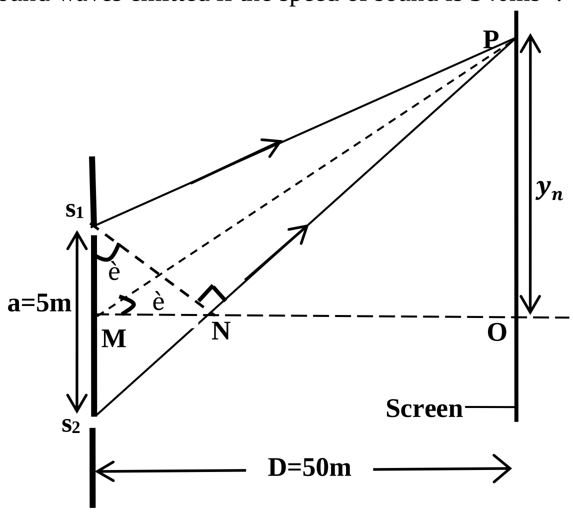
$$y_7 = \frac{(7 - \frac{1}{2})(5.2 \times 10^{-7})(80 \times 10^{-2})}{7.94 \times 10^{-4}m}$$

$$\boldsymbol{y_7} = \mathbf{3.41 \times 10^{-3}m}$$

**3.** Two loud speakers which emit sound of the same frequency are placed as shown below.




A microphone moved along the line PQ detects intensity maxima at regular intervals of 3.4m. Find the frequency of sound waves emitted if the speed of sound is $340ms^{-1}$.




Path difference,$= S_2N$

$$= S_2N = a\sin è \approx a\tan è = \frac{ay_n}{D}$$

For constructive interference, Path difference, $\Delta = \boldsymbol{n}$ë

$\Leftrightarrow \frac{ay_n}{D} = n\lambda$

Separation of maxima;

$\Delta y = \boldsymbol{y_{(n+1)}} - \boldsymbol{y_n}$

$\Delta y = \frac{ëD}{a}$

Thus from;

$\Delta y = \frac{ëD}{a}$

$3.4 = \frac{ë(50)}{5}$

$\underline{ë \;\; = 0.34\,m}$

∴Frequency, $f = \frac{v}{ë} \Leftrightarrow f = \frac{340}{0.34} \Leftrightarrow \underline{f = 1000\text{Hz}}$

**Example:**

**Qn: 1.** Two slits are illuminated by light of two wavelength, one of which is 600nm and whose $4^{th}$ dark fringe coincides with the fifth bright fringe for the other wave length. Find the $2^{nd}$wavelength on a screen.

**Solution:**

For a dark fringe;$\Delta = \left(m + \frac{1}{2}\right)ë$

For a Bright fringe; $\Delta = më$

For coincidence, or overlap; the two path differences are equal.

$$\left(m + \frac{1}{2}\right)ë_1 = më_2$$

$$\left(4 + \frac{1}{2}\right)ë_1 = 5ë_2$$

$$\frac{7}{2}ë_1 = 5ë_2$$

$$ë_2 = \frac{7}{10}ë_1$$

$$ë_2 = \frac{7}{10}(600)$$

$$\underline{ë_2 = 420\text{nm}}$$

**<u>Exercise</u>**

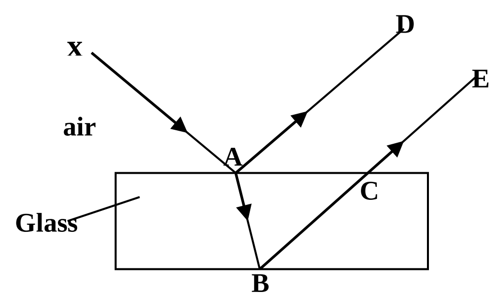
**Qn: 2.**See UNEB, P510/2, 1998 No.3 (b)

**Qn: 3.** See UNEB, P510/2,2001No.4

**Qn: 4**. See UNEB, P510/2, 2002 No.3 (iii)

(ii) **<u>DIVISION OF AMPLITUDE</u>**.E.g air wedge.

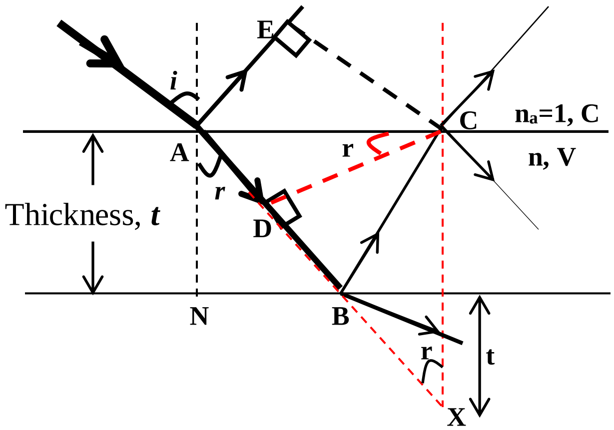
Division of amplitude is a method of producing two coherent sources, where the coherent waves originate from the same point on a wave front, each waves having part of amplitude of the original waves.




- ❖ Part of the energy incident at A is reflected and part is refracted as AB.
- ❖ At A, there is division of intensity. The intensity is proportional to the square of the amplitude.($\boldsymbol{I}$á$\boldsymbol{A^2}$).
- ❖ Therefore, division of amplitude is said to occur at A.

☞ **<u>Parallel – sided thin films.</u>**

Consider monochromatic light incident in air at an angle, i, on a parallel – sided thin film of a material of refractive index, n and thickness, t.




Light from thunder goes successive partial reflection and refraction at the upper and lower surfaces of the film.

The optical Path difference between the rays reflected at A and those reflected from at B and transmitted at C is;

$\text{Path difference, ABC} = \text{n(DB + BC)} = \text{n(DB + BX)} = \text{nDX},$

$\text{Path. Diff. ABC}, \Delta = \text{nDX} \dots\dots\dots\dots.(i),$
From Triangle, DXC,
$\cos r = \dfrac{DX}{2t}$
$\text{DX} = 2\text{t}\cos r \dots\dots\dots\dots\dots\dots\dots\dots.(ii),$
From equations (i) and (ii), we get;
$\text{Path. Diff. ABC}, \Delta \;= \text{nDX}$
$\qquad\qquad\qquad\quad = \text{n}(2\text{t}\cos r)$
$\mathbf{Path.\,Diff.\,ABC}, \Delta = \mathbf{2nt\cos}\boldsymbol{r}$

<u>Alternatively</u>
$\text{Path difference, ABC} = \text{n(AB + BC)} - \text{AE}$
CE and CD represent reflected and refracted wave fronts. Hence the path AE in air is equivalent to the path AD in the film. i.e AE = nAD.
$\text{Path difference, ABC} = \text{n(AB + BC)} - \text{nAD}$
$\text{Path difference, ABC} = \text{n(AB + BC} - \text{AD)},$

By laws of reflection**, AB = BC.**
From Triangle, ANB,

$$\cos \text{r} = \frac{\text{AN}}{\text{AB}} \Leftrightarrow \cos \text{r} = \frac{\text{t}}{\text{AB}} \Leftrightarrow \text{AB}\cos \text{r} = \text{t} \Leftrightarrow \text{AB} = \frac{\text{t}}{\cos \text{r}}$$

$$\sin \text{r} = \frac{\text{NB}}{\text{AB}} \Leftrightarrow \sin \text{r} = \frac{\text{NB}}{\left(\frac{\text{t}}{\cos \text{r}}\right)} \Leftrightarrow \sin \text{r}\left(\frac{\text{t}}{\cos \text{r}}\right) = \text{NB} \Leftrightarrow \text{NB} = \text{t.tanr}$$

**AC = 2NB = 2 t.tanr**

From Triangle, ADC,

$$\sin r = \frac{AD}{AC} \Leftrightarrow \sin r = \frac{AD}{2t\ tanr} \Leftrightarrow AD = 2t.sinr.tanr$$

$\text{Path difference, ABC} = \text{n(2AB} - \text{AD)}$
$\qquad\qquad\qquad\qquad\quad = \text{n}(2\text{t}\sec r - 2\text{t}\tan r\sin r),$

$$= 2\text{nt}\left(\frac{1}{\cos r} - \frac{\sin^2 r}{\cos r}\right)$$

$$= 2\text{nt}\left[\frac{1-(1-\cos^2 r)}{\cos r}\right]$$

$\mathbf{Path.\,Diff.\,ABC}, \Delta \quad = \mathbf{2nt\cos}\boldsymbol{r}$

But there is a phase change of $180^0$ (ð $radians$) at A equivalent path difference of $\frac{\text{ë}}{2}$ **,** since reflection occurs at a denser medium.

Thus, for constructive interference,
Path difference + $\frac{\text{ë}}{2} = \boldsymbol{n}\text{ë}$ ; $\boldsymbol{For\ n} = \mathbf{0, 1, 2, 3}, \dots\dots\dots$

$$2nt\cos r + \frac{\text{ë}}{\mathbf{2}} = \boldsymbol{n}\text{ë}\ ;$$

For destructive interference,
Path difference + $\frac{\text{ë}}{2} = (\boldsymbol{n} + \frac{1}{2})\text{ë}$ ; $\boldsymbol{For\ n} = \mathbf{0, 1, 2, 3}, \dots\dots\dots$

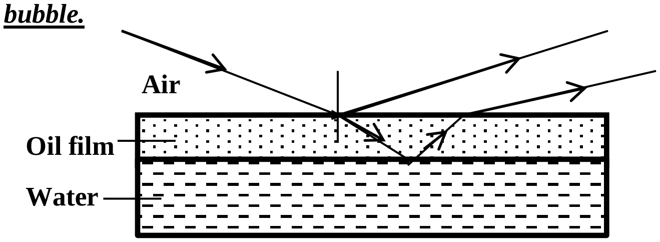
$$2nt\cos r + \frac{\text{ë}}{\mathbf{2}} = (\boldsymbol{n} + \frac{\mathbf{1}}{\mathbf{2}})\text{ë}\ ;$$

**For normal incidence, i, = 0 and r = o; hence,**

$\boldsymbol{2nt} = \left(\boldsymbol{n} - \dfrac{\mathbf{1}}{\mathbf{2}}\right)\text{ë}$ ; For a bright fringe

$\boldsymbol{2nt} = \boldsymbol{n}\text{ë}$ ; For a dark fringe

**Everyday examples of interference.**
***<u>The colours of the oilfilm on water and the colours of a soap bubble.</u>***




- These colours are due to interference between two wave trains, one reflected from the surface of the oil, and the other from the oil – water interface.
- When interference, gives constructive interference for light of one wave length, the corresponding colour is seen in the oil-film.
- The path difference varies with the thickness of the film and the angle of viewing $\left[\ i.e.\quad 2nt\cos r + \frac{\text{ë}}{2} = \boldsymbol{n}\text{ë}\ \right]$

This also affects the colour produced at any one part.

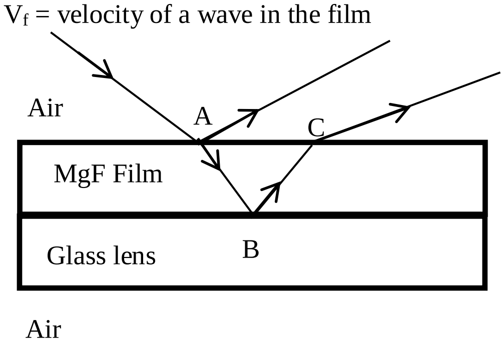
$V_{air} \;\; = f\text{ë}_{air.}$.......................... (i)
$V_{film} = f\text{ë}_{film}$……………..... (ii)

Eqn (i) ÷Eqn (i);

$$\frac{V_a}{V_f} = \frac{f\text{ë}_{air.}}{f\text{ë}_{film}}$$

$$\boldsymbol{\frac{V_a}{V_f}} = \frac{\text{ë}_{\boldsymbol{air.}}}{\text{ë}_{\boldsymbol{film}}} = \boldsymbol{n_{film}}$$

**Example:**
A non reflecting coated lens has a thin film of magnesium fluoride (n =1.25) deposited on the surface of glass (n = 1.56). What should be the thickness of the film in order that green light of wavelength 500nm in air will not be reflected by the coating?
Solution:
Let $V_a$ = velocity of a wave in air
$V_f$ = velocity of a wave in the film




The film produces destructive interference for only one wave length, ë. There is a phase change of ð $rads$ at A and B.

$$\boldsymbol{2n_f}t + \frac{\text{ë}}{2} + \frac{\text{ë}}{2} = \left(\boldsymbol{m} + \boldsymbol{\frac{1}{2}}\right)\text{ë}$$

$$2n_f t = \left(m - \frac{1}{2}\right)\text{ë}$$

Form=1,

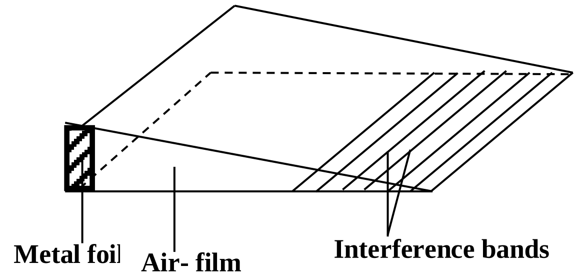
$$2(1.25)t = \left(1 - \frac{1}{2}\right)(500 \times 10^{-9})$$

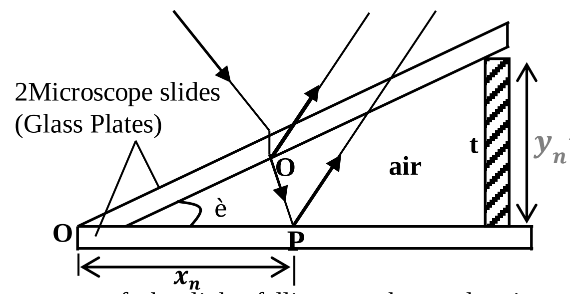
$$t = 100 \times 10^{-9}m \quad = 1.0 \times 10^{-7}m$$

**Qn:** A liquid film of uniform thickness is just thick enough to cause maximum reflection of light of wave length 560nm at normal incidence. If the refractive index of the film is 1.40, find the thickness of the film. (Ans: t = 100nm).

## ☞ The air - wedge thin films.

The air wedge produces interference effects by division of amplitude.
The air wedge is made by clamping two microscope slides (or glass slides) at one end and separating them by a thin sheet of material of thickness, t at the other end.
Monochromatic light is incident almost normally at the glass plate.



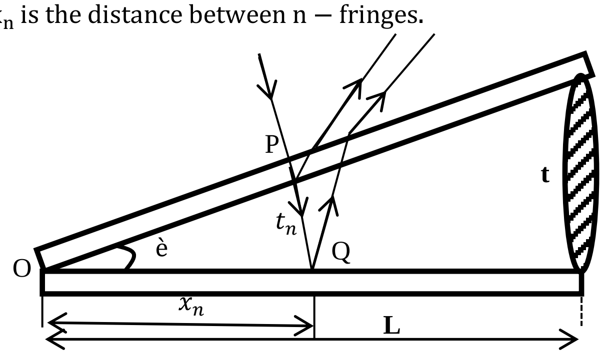

Here some of the light falling on the wedge is reflected upwards from the bottom surface of the top slide i.e at O and the rest which is transmitted through the air wedge is reflected upwards from the top surface of the bottom slide i.e at P so that the amplitude of the wave is divided into two parts.

The two waves from O and P originate from the same point, hence are coherent. Therefore, when brought together (by the eye or microscope), they interfere to form interference patterns.

At O where the path difference is zero, we would expect a bright band, but a dark band is formed instead. This is because, at P, light is reflected from a denser medium, hence it suffers a phase change of $180^0$ or ð $radians$ equivalent to a path difference of $\frac{\text{ë}}{2}$. Hence a crest is reflected as a trough and a trough as a crest.

**Type of fringes formed.**
Equally spaced, alternating dark and bright fringes which are parallel to the edge of contact of the wedge are formed if the surfaces of the plates were flat.

**Fringe Spacing, *x*.**
Let L be the length of each slide,
$\theta$, is the small wadge angle in radians,
$x_n$ is the distance between n − fringes.



The geometrical path difference between the waves reflected from P and Q is 2t.
i.e**Path difference, = t-(-t) = 2t**.

Assuming the fringe covered a distance, L;

$$\tan è = \frac{t}{\mathrm{L}} \Leftrightarrow \boldsymbol{t} = \mathbf{L}\,\mathbf{tan}\, è \ldots\ldots\ldots\ldots.\ldots\ldots..(\mathrm{i})$$

From triangle OPQ;

$$\tan è = \frac{t_n}{x_n} \Leftrightarrow \boldsymbol{t} = \boldsymbol{t_n} = \boldsymbol{x_n}\,\mathbf{tan}\, è \ldots\ldots\ldots..(\mathrm{ii})$$

For constructive interference, (Bright fringe)

$$2t + \frac{ë}{\mathbf{2}} = \boldsymbol{n}ë\ ;$$

$$2x_n \tan è + \frac{ë}{2} = në$$

$$x_n = \frac{\left(n - \frac{1}{2}\right)ë}{2\tan è}$$

For destructive interference, (Dark fringe)

$$2t + \frac{ë}{\mathbf{2}} = (\boldsymbol{n} + \frac{\mathbf{1}}{\mathbf{2}})ë\ ;$$

$$2x_n \tan è + \frac{ë}{2} = (\mathrm{n} + \frac{1}{2})\lambda$$

$$x_n = \frac{në}{2\tan è}$$

<u>The fringe spacing$x$;</u>
Considering a dark fringe;

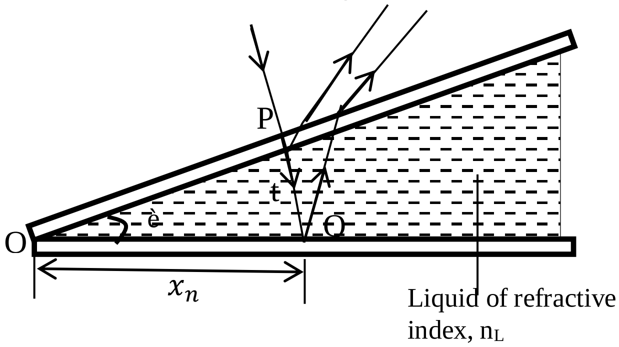
$$x = x_{n+1} - x_n$$

$$= \frac{(n+1)ë}{2\tan è} - \frac{në}{2\tan è}$$

$$= \frac{ë}{2\tan è}(n + 1 - n)$$

$$\boldsymbol{x} = \frac{ë}{\mathbf{2}\,\mathbf{tan}\, è} : \quad \text{Note; Also } \boldsymbol{x} = \frac{\boldsymbol{x_n}}{\boldsymbol{n}}$$

<u>Effect of on the fringe spacing when a liquid of refractive index, $n_L$ is introduced in the wedge.</u>




In this case,
Optical path = $\mathbf{n_L}$(Path difference)

$$\text{Optical path } = n_L(2t)$$
$$= 2n_L t$$

$$\textbf{Optical path} = \mathbf{2}\boldsymbol{n_L x_n}\,\mathbf{tan}\, è$$

Thus if we consider a dark fringe,

$$2t + \frac{ë}{2} = (n + \frac{1}{2})ë$$

$$2n_L x_n \tan è + \frac{ë}{2} = (n + \frac{1}{2})ë$$

$$x_n = \frac{në}{2n_L \tan è}$$

Hence the fringe spacing, x is given by;

$$\boldsymbol{x} = \frac{ë}{\mathbf{2}\boldsymbol{n_L}\,\mathbf{tan}\, è}$$

Therefore, when a liquid is introduced, the fringe spacing is reduced by a fraction of the refractive index of the liquid, $n_L$.

**<u>Alternatively</u>**
From the fringe spacing in an air wedge,

$$x_{air} = \frac{ë_{air}}{2\tan è}$$

$$x_{Liquid} = \frac{ë_{Liquid}}{2\tan è} \ldots\ldots\ldots\ldots\ldots(i)$$

From the definition of refractive index,

$$n_L = \frac{C}{V_L}$$

Where,$\mathbf{n_L}$is refractive index of the liqid,**C** = velocity of the wave in air, and $\mathbf{V_L}$ is the velocity of the wave in the liquid.

$$n_{Liquid} = \frac{fë_{air}}{fë_{Liquid}}$$

$$ë_{Liquid} = \frac{ë_{air}}{n_{Liquid}} \ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

From Equations (i) and (ii), substituting for $ë_{Liquid}$ in (i) gives;

$$x_{Liquid} = ë_{Liquid}\left(\frac{1}{2\tan è}\right)$$

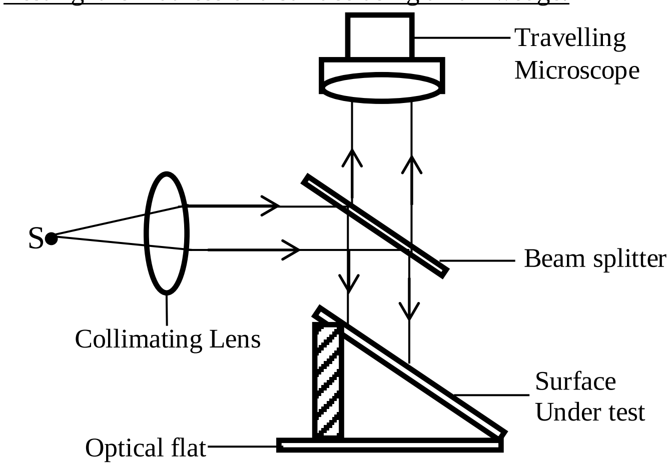
$$x_{Liquid} = \frac{ë_{air}}{n_{Liquid}}\left(\frac{1}{2\tan è}\right)$$

$$\boldsymbol{x_{Liquid}} = \frac{\boldsymbol{ë_{air}}}{\mathbf{2}\boldsymbol{n_{Liquid}}\,\mathbf{tan}\, è}$$

<u>Effect of using white light instead of monochromatic light.</u>
- ♦ Coloured fringes are observed. The blue fringe is nearest to the edge of contact while the red fringe is furthest from the point of contact of the glass plates.
- ♦ After the first red fringe, the next coloured fringes overlap.

- ♦ At very large distances from the edge of contact, the fringes are so much out of phase and no fringes are observed but instead white illumination.

Testing the Flatness of a surface using an air wedge.

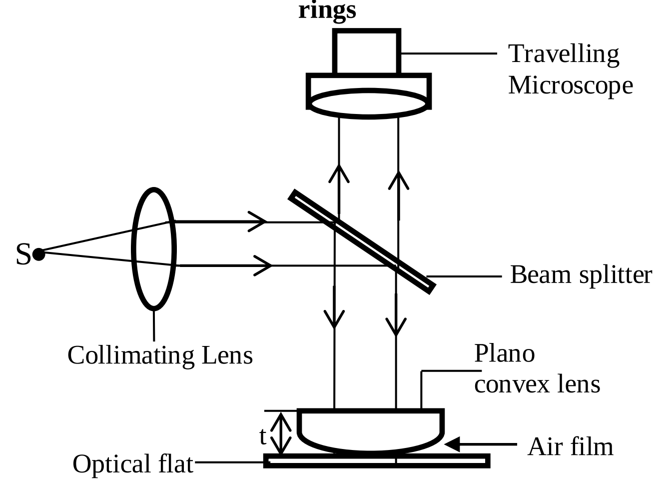



- Monochromatic light from a source S at the principle focus of the collimating lens is made to strike the beam splitter.
- Light is reflected and falls almost normally onto the air wedge. The reflected light forms interference patterns which are observed.
- If the surface under test was flat, equally spaced fringes, parallel to the edge of contact of the wedge are observed.
- If the surface was not flat, an uneven (or irregular) fringe pattern is observed.

☞ **Newton's rings:**

Newton's rings are a system of concentric alternating dark and bright interference fringes produced by division of amplitude.

**Experimental setup for observing Newton's rings**




A parallel beam of monochromatic light from a sodium lamp placed at the principal focus of the collimating lens is reflected by a beam splitter to fall normally on an air film between the convex lens of long focal length and an optical flat (glass plate). The layer of air trapped forms an air wedge.

Interference occurs between light reflected from the lower surface of the lens and the supper surface of the glass plate (or optical flat)

A series of alternate dark and bright fringes is observed using a travelling microscope. These interference patterns (fringes) are known as Newton's rings

The circular fringes have increasing radius and are concentric with the point of contact of the lens and the optical flat.

At the point of contact (center), the geometric path difference between the two wave trains is zero. There is a dark spot at the center due to the $180^0$ phase change which occurs when light is reflected at an optically dense medium (optical flat)

Therefore;

For constructive interference (Bright fringe)

Path difference $+\frac{\text{ë}}{2} = n\text{ë}$

$$2t + \frac{\text{ë}}{2} = n\text{ë}$$

For Destructive interference (Dark fringe)

Path difference $+\frac{\text{ë}}{2} = (n+\frac{1}{2})\text{ë}$

$$2t + \frac{\text{ë}}{2} = (n + \frac{1}{2})\text{ë}$$

**NOTE:** The Newton's rings can be used to determine the flatness of a lens. If the lens is not properly grinded i.e. imperfect, the rings observed are distorted.i.e.they are not perfectly circular.

If **R** is the radius of curvature of the lens, the radius of the $n^{th}$ – dark or bright fringe (**$r_n$**) is calculated from;

$r_n^2 = R(2t)$

$\Rightarrow r_n^2 = R(n\text{ë})$……………...For a dark fringe

$r_n^2 = R(n - \frac{1}{2})\text{ë}$…………..For a bright fringe

**Uses or applications of interference**

(i) Testing the flatness of optical surfaces.

The surface under test is made in form of an air wedge with a standard glass surface.

Any irregularities in the surface will show up as irregularities in what should have been a parallel, equally spaced straight set of fringes.

(ii) Blooming (Creating non-reflecting glasses).

This is done to reduce the amount of light reflected at the surface, by coating the surface with magnesium fluoride.

(iii) Measuring wave length

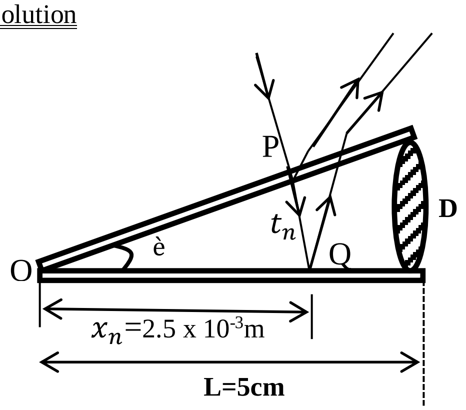
(iv) Pulsing of the picture on a television.

**Examples:**

**1.** An air wedge is formed by placing two glass slides of length 5.0cm in contact at one end and a wire at the other end. Monochromatic light of wavelength 550nm is made incident almost normally on the upper plate. Viewing from vertically above, 10 dark fringes are observed to occupy a distance of 2.5mm.

(i) Explain how the fringes are formed
(ii) Determine the diameter of the wire
(iii) Find the total number of fringes observed
(iv) What is the fringe spacing in the interference pattern observed?

Solution




Given: $x_n$=2.5mm = 2.5 x 10$^{-3}$m

$$ë = 550nm = 550 \times 10^{-9}m$$

n = 10

(i) See notes

(ii) From the distance of the n$^{th}$ dark fringe;

From the diagram;

$$\tan è = \frac{D}{x_n} = \frac{D}{2.5 \text{ x } 10^{-3}} \ldots\ldots\ldots\ldots (i)$$

For a dark fringe in an air wedge,

$$2t + \frac{\lambda}{2} = (n + \frac{1}{2})\lambda$$

$$2D = n\lambda$$

$$D = \frac{10 \times 550 \times 10^{-9}}{2} = 2750 \times 10^{-9} = 2.75 \times 10^{-6}\text{m}$$

(iii)

From the distance of the n$^{th}$ dark fringe;

☞ Since 10 dark fringes were observed, also 10 bright fringes were seen. Thus a total of 20 fringes were observed.

(iv) From the fringe spacing;

$$x = \frac{x_n}{n}$$

$$x = \frac{2.5 \times 10^{-3}}{10} = 2.5 \times 10^{-4}\text{m}$$

Alternatively; From;

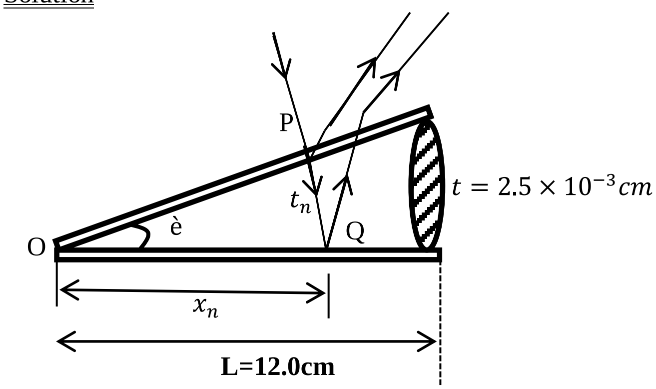
$$x = \frac{ë}{2\tan è} = \frac{ë}{2\left(\frac{D}{x_n}\right)}$$

$$x = \frac{550 \times 10^{-9}}{2\left(\frac{2.75\times10^{-6}}{2.5 \text{ x } 10^{-2}}\right)} = 2.5 \times 10^{-4}\text{m}$$

**2.** Two glass plates 12.0cm long are in contact at one edge and are separated by a piece of metal foil $2.5 \times 10^{-3}m$ thick. When the plates are illuminated normally by light of wave length 500nm, a system of fringes is formed. Find the;

(i) Fringe separation
(ii) Number of dark fringes formed

Solution




Given: $t$ = 2.5 x 10$^{-}$3cm

$$ë = 500nm = 500 \times 10^{-9}\text{m}$$

(i) From the fringe separation;

$$x = \frac{ë}{2\tan è} = \frac{ë}{2\left(\frac{D}{L}\right)}$$

$$x = \frac{500 \times 10^{-9}}{2\left(\frac{2.5\times10^{-3}\times10^{-2}}{12.0 \text{ x } 10^{-2}}\right)} = 1.2 \times 10^{-3}\text{m}$$

(ii) Number of fringes formed;

For a dark fringe to be observed,

$$2t + \frac{\lambda}{2} = (n + \frac{1}{2})\lambda$$

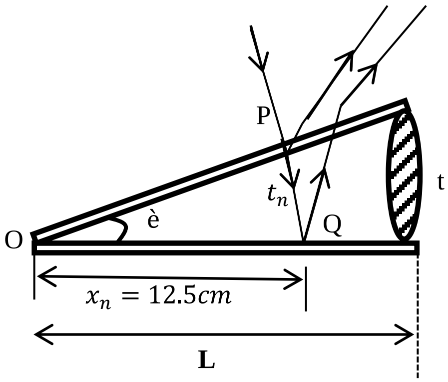
$$2t = n\lambda$$

$$n = \frac{2t}{\lambda} = \frac{2\times2.5\times10^{-3}\times10^{-2}}{500\times10^{-9}} = 100 \text{ DarkFringes}$$

**3.** Two glass plates in contact at one end are separated by a metal foil 12.50cm from the line of contact to form an air wedge. When the air wedge is illuminated normally by light of wave length 5.4 x 10$^{-7}$m, interference fringes of separation 1.5mm are found on reflection. Find the;

(i) Thickness of the metal foil.
(ii) New fringe separation if the air wedge is filled with a liquid of refractive index 1.33.

Solution




(i) Given: $x_n$= 12.5 x $10^{-2}$m, x=1.5mm 15 x $10^{-3}$m
$\text{ë} = 5.4 \times 10^{-7}\text{m}$

From the fringe separation;

$$x = \frac{\text{ë}}{2\tan\text{è}} = \frac{\text{ë}}{2\left(\frac{t}{x_n}\right)} = \frac{\text{ë}x_n}{2t}$$

$$t = \frac{\text{ë}x_n}{2x} = \frac{5.4 \times 10^{-7}(12.5 \times 10^{-2})}{2(1.5 \times 10^{-3})} = 2.25 \times 10^{-5}m$$

(ii) When a liquid of refractive index n is introduced, the fringe spacing, x reduces to;

$$x = \frac{\text{ë}}{2n\tan\text{è}} = \frac{\text{ë}}{2n\left(\frac{t}{x_n}\right)} = \frac{\text{ë}x_n}{2nt}$$

$$t = \frac{\text{ë}x_n}{2nx} = \frac{5.4 \times 10^{-7}(12.5 \times 10^{-2})}{2(1.33)(1.5 \times 10^{-3})} = 1.69 \times 10^{-5}m$$

Alternatively:

$$x_{liquid} = \frac{x_{air}}{n_{Liquid}} = \frac{1.69 \times 10^{-5}m}{1.33} = 1.23 \times 10^{-5}$$

**4.** A wire is placed between the edges of 2 flat glass plates which are in contact at the other end. The air wedge shaped air film between the plates is viewed by reflecting monochromatic light of wave length 589nm and is found to be crossed by 35 dark fringes. Calculate the diameter of the wire.
(Ans: D = $1.016 \times 10^{-5}$m)

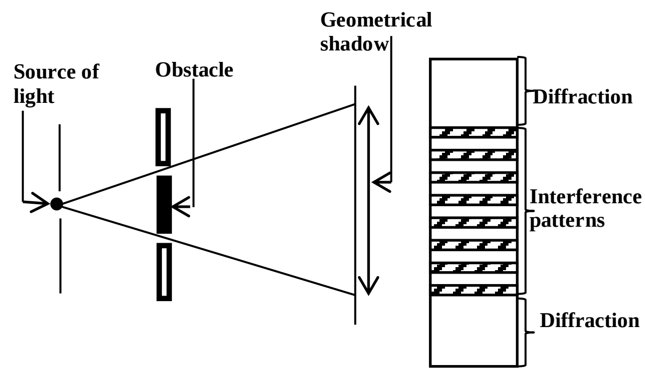
**5.** See UNEB 2000 No.4
See UNEB 2004 No.
See UNEB 2007 No.4 (a) (iii)
See UNEB 2008 No.4 (d)

## (B) DIFFRACTION OF LIGHT

Diffraction is the spreading of light beyond its geometric shadow, leading to interference patterns at the edges of the shadow.

It can also be defined as the spreading of a wave when it passes through a narrow aperture whose dimensions are comparable to the wave length of the wave.

Diffraction is due to superposition of secondary wavelets from a continuous section of a wave front that has been obstructed.

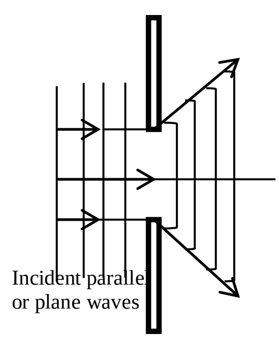



Note: where as interference involves superposition of waves onto different wave fronts, Diffraction involves superposition of waves onto different parts of the same wave front.

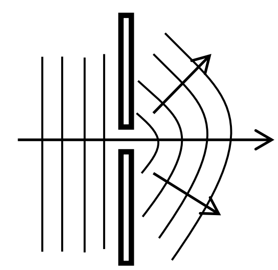
**Conditions necessary for observing diffraction patterns**

- The dimensions or size of the aperture in the obstacle must be of the same order as the wave length of the incident wave.

Usually the smaller the width of the aperture, the greater the spreading of the wave (i.e. the greater the diffraction).
Similarly, the longer the wavelength of the incident wave, the greater the spreading or diffraction.




Wide Gap

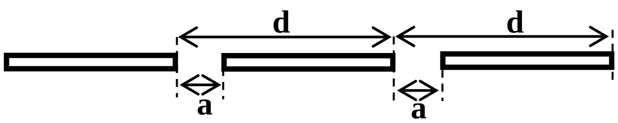


Narrow Gap

**Note:** It is possible for us to hear around corners because, sound waves have longer wave length and therefore can greatly be diffracted even around relatively wider apertures hence allowing sound to spread around the corner.
However, we cannot see around corners because light waves have a much shorter wavelength and there doesn't spread appreciably beyond its geometrical shadows.

**Diffraction grating**

A diffraction grating is a set of many parallel closely spaced, equidistant narrow slits ruled on either a glass plate (For a transmission grating) or a polished metal surface (For a reflection grating).



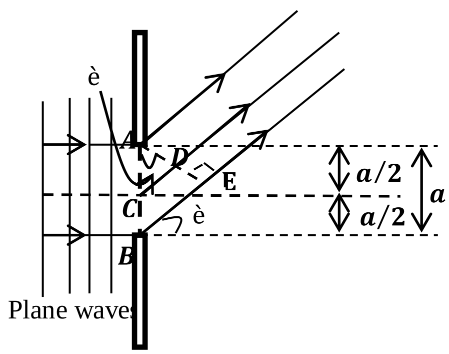

Where a = slit spacing, and d = spacing between adjacent line of the grating.
For a grating of **e- lines per mm**, the spacing between adjacent lines, d, is given by;

$$\mathbf{d} = \frac{\mathbf{1}}{\mathbf{e}}$$

For example, a grating of 600 lines per mm, the lines spacing d is; $\mathrm{d} = \frac{1}{\mathrm{e}} = \frac{1}{600} = 0.00167\mathrm{mm}$.

Diffraction grating is used to produce optical spectra.

**Diffraction on a single slit of width ‘a’**




Single slit

From triangle, ACD:

$$\sin \text{è} = \frac{CD}{AC} = \frac{CD}{a/2} \Leftrightarrow \text{Path difference, CD} = \frac{a}{2}\sin \text{è} \dots\dots (i)$$

From triangle, ABE:

$$\sin \text{è} = \frac{BE}{AB} = \frac{BE}{a} \Leftrightarrow \text{Path difference, BE} = \text{a}\sin \text{è} \dots\dots (ii)$$

The slit is assumed to consist of strips of equal width parallel to the length of the slit.

A central white band with dark bands on either sides is formed with coloured fringes at the edge of each dark band

Consider the first minimum (1st dark band) where there is no light. It will be formed at an angle è to the incident beam if the path difference for the secondary wave lets from the strip just below A and the strip just below C (the midpoint of the slit) is $\frac{\lambda}{2}$ where λ is the wave length of the incident light.

There is destructive interference since a crest from one strip A reaches the observer with a trough from the next strip C
This happens for all pairs of corresponding strips in A, C and CB since the same path difference of $\frac{\lambda}{2}$ exists.

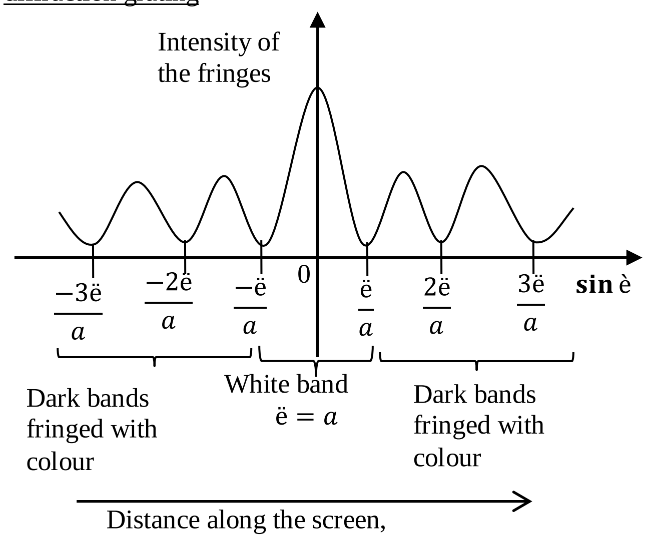
Therefore, there is no light in the direction è when the path difference, CD=$\frac{\lambda}{2}$ or path difference BE =λ
Thus for the first minima, ($1^{st}$ dark band)

Path difference, CD = $^{\text{ë}}/_{2} = ^{\text{á}}/_{2}\, sin\text{è} \Longrightarrow \text{á}\mathbf{sin}\,\text{è} = \text{ë}$

For the ($n^{\text{th}}$ dark band)

Path difference CD=$^{n\text{ë}}/_{2} = ^{a}/_{2}\sin \text{è} \Longrightarrow \text{á}\mathbf{sin}\,\text{è} = \boldsymbol{n}\text{ë}$

<u>Intensity distribution on a screen placed beyond a single slit diffraction grating</u>




At the center of the slit, the path difference is zero. Each fringe has a bright band. Hence the central band is white.

If a red filter is put between alamp and a screen, red & blackbands are seen.

**NOTE:** from $\mathbf{sin}\,\text{è} = {}^{\boldsymbol{n}\text{ë}}/_{\boldsymbol{a}}$

(i) If the slit width **a** is large (wide), sinè and è are very small hence the minima are very close to the central maximum. Most of the light emerging from the slit is in the direction of the incident light hence there is little diffraction. The propagation is almost rectilinear and laws of geometrical optics apply.(α>>λ)

(ii) If the slit width a is reduced say $a \approx$ ë

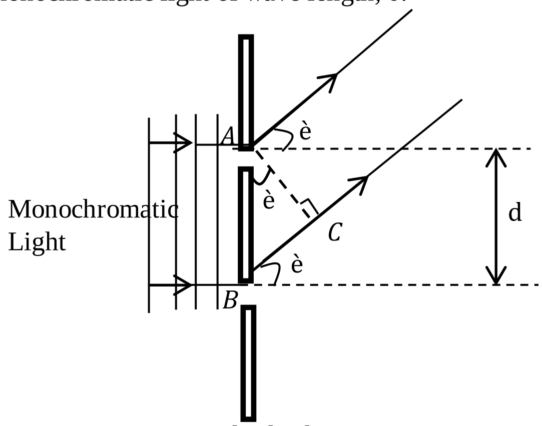
- A diffraction pattern is observed. In this case, the behavior of light requires a wave model.
- A central white band with dark bands on either sides is observed. Each dark band from either side has coloured fringes running fromblue (near the central position) to the red ( far away from the central position ).

(iii) If the slit width is reduced further, á∠ë
The central white band widens and extends well into the geometrical shadow of the slit.(i.e. more diffraction occurs)

(iv) When the slit finally closes, no light passes through and hence, no fringes are observed at all.

**Diffraction on a grating with multiple slits**

Consider a transmission grating illuminated normally by monochromatic light of wave length, ë.




Multiple slits

From triangle, ABC:

$\sin$ è $= \frac{BC}{d} \Leftrightarrow$ Path difference, $\mathbf{BC = d\,sin}$ è ...... $(i)$

Each of the clear spaces A, B act like a narrow slit and diffract the incident light in all the forward directions.

If we consider light diffracted at an angleθto the normal and θis such that light from A is in phase with that from B, then it is also in phase with every other slit since all the slits are equally spaced.

Reinforcement (constructive interferencea0 of the diffracted wavelets occurs in the direction θto form a bright band when the path difference is an integral multiple of the incident wavelength.

Path difference, $\mathbf{BC= n\,\lambda}$ ........................................(ii)

From equation (i) and (ii);

$$\mathbf{d\,sin}\text{ è} = \mathbf{n\,\lambda}$$

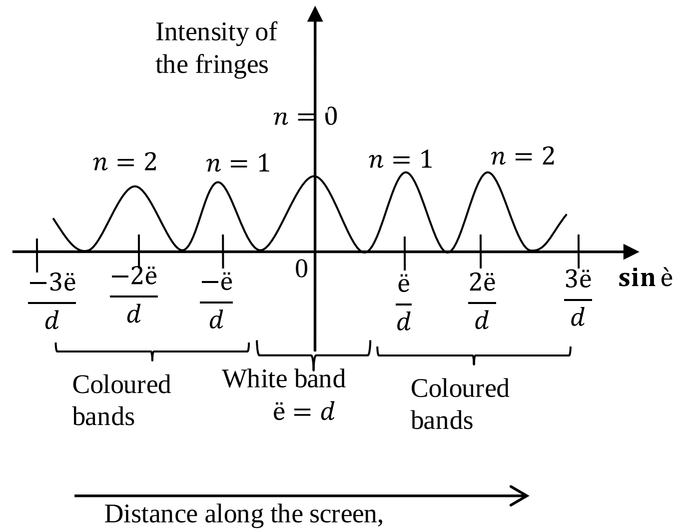
Where n is an integer called the order of diffraction maximum and **n** = 1, 2, 3…

The zeroth order (n = 0) diffraction maximum corresponds to an angular separation (or position) ofθ = 0.Here, the path difference of the diffracted wavelets is zero and the diffraction maximum is white.

For the first order diffraction, n = 1, For the second order diffraction, n = 2, e.t.c.

Intensity distribution on a screen placed beyond a diffraction grating of a large number of slits.

The intensity distribution consists of principle maximums of almost equal intensity just like those in Young's double slit interference.



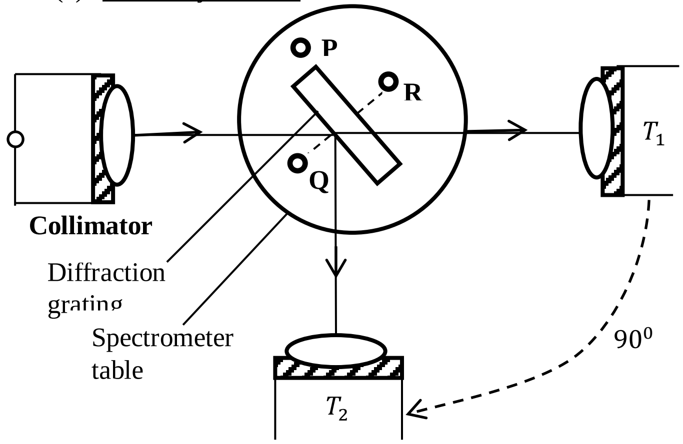

Effects on the diffraction pattern of using a grating with large number of lines ( i. e. large è)

**From sin** è $= {}^{\boldsymbol{n}\text{ë}}/_{\boldsymbol{d}}$

- The grating spacing d reduces
- Sinè and è (position of principal maxima) increases ie the number of subsidiary maxima between each pair of principal maxima increases
- Separation of the diffraction maxima (principal maxima) increases.

**Measurement of wavelength of monochromatic light using a spectrometer and a transmission grating**

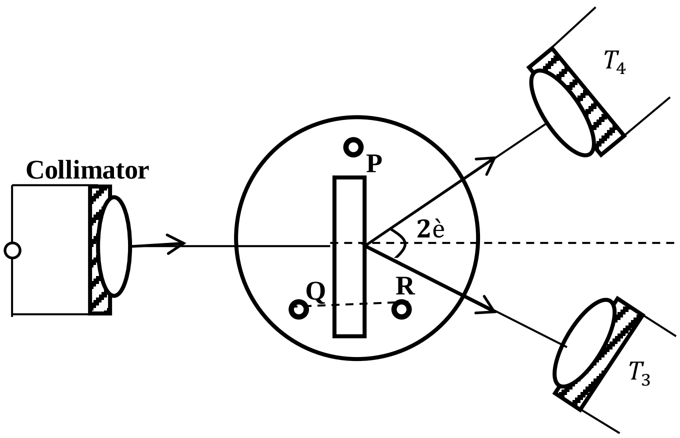
(a) Initial adjustments




- The collimator slit and telescope are adjusted for parallel light ie. Such that the collimation slit is in sharp focus
- The grating is placed on the spectrometer table making $90^0$ with screws Q and R
- The telescope is rotated through exactly $90^0$ from positionT$_1$ and T$_2$.
- The table and the grating are rotated until the image of the collimator slit reflected from the grating is in the center of

the field of view. The plane of the grating is now parallel to the axis of rotation of the telescope

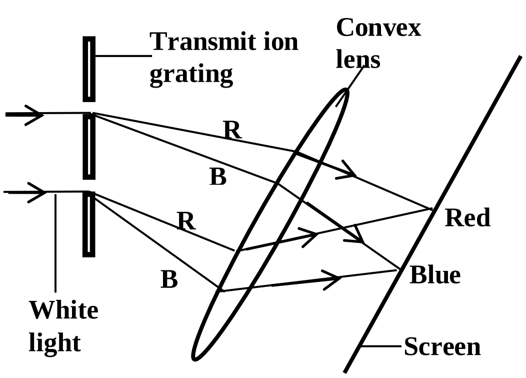
(b) Measurement of wavelength




- The table and the grating are then rotated through exactly $45^0$ such that the incident light falls perpendicularly to the grating
- The telescope is rotated to position $T_3$ where the first order diffraction image is seen. The angle of rotation è from the normal is noted
- The telescope is rotated to the position $T_4$ where the first order diffraction image is seen on the other side of the normal. The angle of rotation è is also noted
- The average rotation $è = {}^{2è}/_{2}$ is calculated and the wave length of the incident light is obtained from $ë = dSinè$

**Formation of a spectrum by a transmission grating**

If white light falls on a transmission grating, different wavelengths are deviated by different amounts and hence travel in different directions




**Why the sky appears blue on a clear clouds day**

- Sunlight consists of many colours of different wavelengths (frequencies), extending from violet to red for visibility light and it travels in a straight line in space as long as northing disturbs it.
- When it passes through the atmosphere (dust, gas-molecules, water droplets e.t.c.) what happens to it depends on the wave length and the size of the particles it hits
- When it hits large particles (e.g. dust & water droplets), it is reflected in a different direction. it still contains all the colours hence it is still white
- when it hits small particles (e.g. gas molecules) small than the visible light, some of it is absorbed & after a while it radiates/ scatters
- The higher frequency (short wavelength) Blues are absorbed more hence are more scattered than the low frequency (low wavelength) red. (Amount scattered $\propto \frac{1}{ë^4}$) this is Rey leigh scattering.
- The absorbed blue light is radiated in different directions. It gets scattered all around the sky hence the appears blue

Uses of diffraction of light

- measurement of wavelength of light using a diffraction grating
- in Holography (making of 3- dimensional images)

Advantages of a diffraction grating over a glass prism in measurement of wave length

(i) A transmission grating leads to direct determination of wave length ë, while use of prism requires a complicated formula

(ii) By using different diffraction orders, n=1,2,3,4 etc an average wave length can be obtained while the prism method yields only one less accurate value

**Example**

1. When monochromatic light of wave length $5.8\times10^{-7}$ m is incident normally on a transmission grating. The second order diffraction line is observed at an angle of $27^0$ Calculate
   (i) The number of lines per cm of the grating
   (ii) The angular deviation of the $3^{rd}$ order image

**Solution**

Given; $ë = 5.8 \times 10^{-7}$m , $è = 27^0$, $n = 2$

**(i)** From ;

$$\text{dsin è} = \text{n}\,\lambda$$

$$\text{d} = \frac{\text{n}\,\lambda}{\text{sin è}}$$

$$\text{d} = \frac{2\times 5.8 \times 10^{-7}}{\sin 27} = 255.5 \times 10^{-6} m$$

The number of lines per unit length is given by;

$$e = \frac{1}{d} = \frac{1}{255.5 \times 10^{-6}} = 3.91 \times 10^{3} \text{lines per m}$$

**(ii)** For the $3^{rd}$ order, n= 3. Thus from;

$$\text{dsin è} = \text{n}\,\lambda$$

$\sin è = \frac{n\lambda}{d}$

$\sin è = \frac{3 \times 5.8 \times 10^{-7}}{255.5 \times 10^{-6}} = 6.81 \times 10^{-3}$

$è = \sin^{-1}(6.81 \times 10^{-3})$

$è = 0.39^0$

**Example**

2. A diffraction grating has 550 lines per mm. when illuminated by light; the angle between the central maxima & the 1st maxima is 19.1$^{0.}$ Find the
   (i) Wavelength of light used
   (ii) Number of diffraction maxima obtainable

**Solution**

Given; $e = \frac{550 lines}{1mm} = \frac{550 lines}{1 \times 10^3 m} =$, $è = 19.1^0$, $n = 1$, For the first order.

$d = \frac{1}{e} = \frac{1}{550} = \frac{1 \times 10^{-3}}{550} = 1.818 \times 10^{-6}$m

**(i)** From ;

$d\sin è = n\lambda$

$$\lambda = \frac{d\sin è}{n} = \frac{1.818 \times 10^{-6}}{\sin 27} = 5.95 \times 10^{-7} m$$

**(ii)** $d(\sin è)_{max} = n_{max} ë$

$$d = n_{max} ë$$

$$n_{max} = \frac{d}{ë} = \frac{1.818 \times 10^{-6}}{5.95 \times 10^{-7}} = 3.1 \approx 3$$

Therefore, a maximum of 7maxima are obtained. That is 3 on either sides of the zero order maxima and the zero order itself.

**Example**

**3.** A source emitting light of two wavelengths is viewed through a grating spectrometer set at normal incidence. When the telescope is set at an angle of 20$^0$to the incident direction, the second order maxima for one wavelength is seen superposedon the third order maximum for the other wavelength. If the shorter wavelength is 400nm, Calculate
   (i) The longer wavelength and the number of lines per meter in the grating
   (ii) The angle at which superposition of the two orders can be seen using this source.

**Solution**

(i)

Let $ë_1$ and $ë_2$ be the wave length

Let $n_1$ and $n_2$ be the respective orders.

When $è = 20^0, n_1 = 2, n_2 = 3$

Then From;

$d\sin è = n\lambda$

$\Rightarrow d\sin 20 = 2ë_1 \dots\dots\dots\dots\dots\dots\dots.(i)$

$d\sin 20 = 3ë_2 \dots\dots\dots\dots\dots\dots\dots.(ii)$

Equation (i)÷ Equation (ii) gives

$2ë_1 = 3ë_2 \Leftrightarrow ë_1 = \frac{3}{2}ë_2$

Thus $ë_1 > ë_2$ hence $ë_2 = 400$nm

$ë_1 = \frac{3}{2}ë_2 \Leftrightarrow ë_1 = \frac{3}{2}(400) = 600$nm

$ë_1 = 600\text{nm} = 600 \times 10^{-9}\text{m} = 6 \times 10^{-7}\text{m}$

From Equation (i)

$d\sin 20 = 2ë_1$

$d\sin 20 = 2(600 \times 10^{-9})$

$d = \frac{2(600 \times 10^{-9})}{\sin 20}$

$d = 3.51 \times 10^{-6}$m

Thus the number of lines per metre, **e** is given by;

$e = \frac{1}{d}$

$e = \frac{1}{3.51 \times 10^{-6}}$

$e = 2.85 \times 10^5$lines per metre.

(ii)

For overlap or superposition, the path differences are equal.

Then From;

$d\sin è = n\lambda$

$\Rightarrow d\sin 20 = n_1ë_1 \dots\dots\dots\dots\dots\dots\dots.(i)$

$d\sin 20 = n_2ë_2 \dots\dots\dots\dots\dots\dots\dots.(ii)$

From Equations (i) and (ii)

$\Rightarrow n_1ë_1 = n_2ë_2$

$n_1(600 \times 10^{-9}) = n_2(400 \times 10^{-9})$

$3n_1 = 2n_2$

When$n_1 = 2, and n_2 = 3,$

For the second and third order maximas to overlap

$n_1 = 2 \times 2, = 4$ $and n_2 = 3 \times 2 = 6$

*For* $n_1 = 4$

$\Rightarrow d\sin 20 = n_1ë_1$

$\Rightarrow 3.51 \times 10^{-6}\sin è = 4(6 \times 10^{-7})$

$\sin è = 0.686$

$è = 43.3^0$

**Example**

**4.** A transmission grating of $5 \times 10^5$lines per meter is illuminated with light of wavelengths 580nm and 590nm. Calculate the:
   **(i)** Highest order spectrum observed
   **(ii)** Angular separation of the two waves in the second order spectrum

***Solution***

**(i)** $d \sin è = në$,

$n_{max} \leq \frac{d}{ëmin}$ but; $d = \frac{1}{5\times10^{-5}}$m;

$n_{max} \leq \frac{2.0\times10^{-6}}{580\times10^{-9}} = 3.4$

Thus the highest order spectrum is third order.

**(ii)** From $d \sin è = në$

$2 \times 10^{-6} \sin è_1 = 2(580 \times 10^{-9})$

$è_1 = \sin^{-1}\left(\frac{2(580 \times 10^{-9})}{2 \times 10^{-6}}\right)$

$\underline{\underline{è_1 = 35.5^0}}$

Also

$2 \times 10^{-6} \sin è_2 = 2(590 \times 10^{-9})$

$è_2 = \sin^{-1}\left(\frac{2(590 \times 10^{-9})}{2 \times 10^{-6}}\right)$

$\underline{\underline{è_1 = 36.2^0}}$

Thus the angular separation;

$\theta = \theta_2 - \theta_1$

$\theta = 36.2^0 - 36.2^0$

$\theta = 36.2^0 - 35.5^0$

$\underline{\underline{\theta = 0.7^0}}$

Exercise

***Question One***

A beam of monochromatic light of wavelength 630nm passes through a slit0.070cm wide and produces a diffraction pattern on a screen 0.80cm away. Find the distance between the third dark fringes from the center of the diffraction pattern

***Question Two***

A diffraction grating of $8 \times 10^5$ lines per meter is used to resolve two lines of 589nm & 589.6nm of sodium light. Calculate the

(i) angular separation in the 1st order and the 2nd order spectrum
(ii) Mention the advantages & disadvantages of using the 2nd order

***Question Three***

See UGADES 2012 Joint Mock
See UNEB 2003 No. 4(d).
2002 No.4(c)
2006 No.4 (d)

## (C) POLARISATION OF LIGHT

Polarization is the orientation in oscillation of a wave in space.

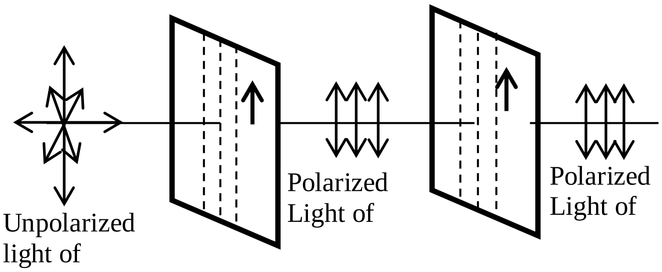
Polarized light(plane polarized light). This is the light which is transmitted by vibrations in only one particular plane. Light in other directions is absorbed by the medium.

**Production of plane polarized light:**

**Note:** Longitudinal waves such assound waves cannot be polarized because its vibrations are already in only one plane.

**(i) Selective absorption** (by using a Polaroid)

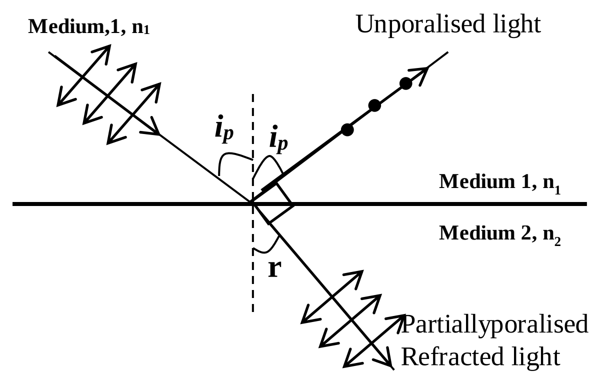
- A Polaroid is a material which transmits only those components of vibrations of only light which is incident on it in a particular plane (or direction)




- A sheet of polaroid is placed with its plane perpendicular to the direction of travel of light rays
- Only light vibrations of a particular orientation to the axes of the crystals(of which the Polaroid is made) are transmitted

**(ii) By reflection**.

- Un polarized light is made incident on a transparent medium (eg. Water or glass) at an angle $i_p$such that the reflected light is perpendicular to the refracted light.




From the diagram;

$\boldsymbol{n_1 \sin i_p = n_2 \sin r}$

$But, r + 90 + i_p = 180^0$

$r = 180^0 - (90 + i_p)$

$r = 90^0 - i_p$

$Thus, n_1 \sin i_p = n_2 \sin(90^0 - i_p)$

$n_1 \sin i_p = n_2 \cos i_p$

$\boldsymbol{\tan i_p = \frac{n_2}{n_1}}$

If medium 1 is air, then $n_1=1$ and thus the above expression become: $\boldsymbol{\tan i_p = n}$.

This equation is known as **Brewster's Law**$\boldsymbol{i_p}$ is an angle at which the reflected ray is completely or totally plane polarized and it's called the angle of polarization.

**Example (UNEB 2010)**

A parallel beam of un polarized light incident on atransparent medium of refractive index 1.62 is reflected as plane polarized light. Calculate the angle of incidence in air and the angle of refraction

**Solution**

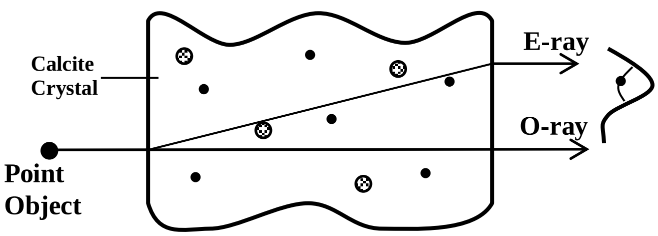
$\boldsymbol{From};\ \tan \boldsymbol{i_p} = \boldsymbol{n}.$

$$\tan i_p = 1.62$$
$$i_p = \tan^{-1}(1.62)$$
$$i_p = 58.3^0$$

$$i_p + 90^0 + r = 180^0$$
$$58.3^0 + 90^0 + r = 180^0$$
$$r = 31.7^0$$

**(iii) Double refraction**

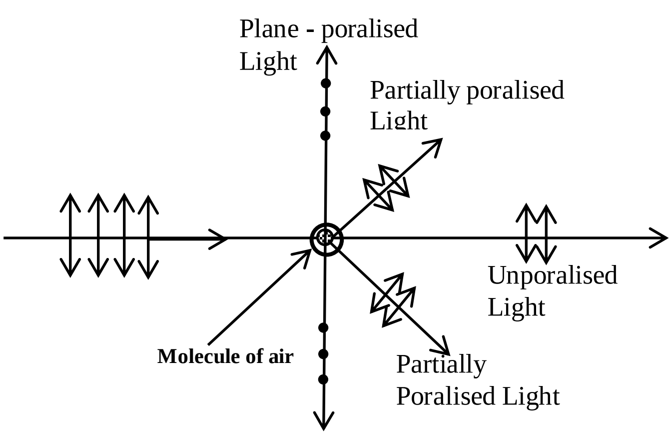
- When un-polarized light is incident on a crystal of calcite (a form of $CaCO_3$), it is split into two rays and these are the ordinary & the extra ordinary rays (E-rays). These are perpendicular to each other and are plane polarized



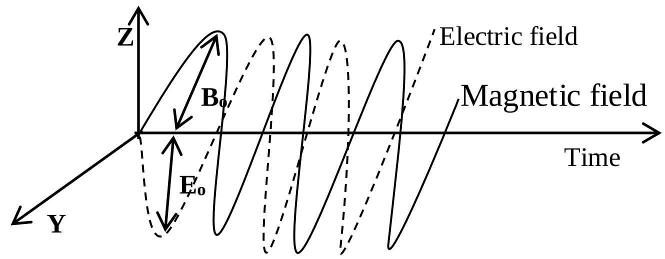

- The ordinary ray obeys Snell's law but the extra ordinary ray does not because it travels with different speeds in different directions in the calcite crystal
- If the point object is viewed through a calcite crystal, the two images are seen. This phenomenon is called double refraction.

**(iv) By scattering**

When un-polarized light is incident on a molecule of air part of it is scattered. The scattered light in a direction perpendicular to the incident light is totally plane polarized.




<u>Sketch of the time variation of electric and magnetic field vectors in plane polarized light.</u>




<u>**Uses and applications of polarized light**</u>

(i) Saccharimetry i.e. measuring the concentration of optically active solutions e.g. sugar solution

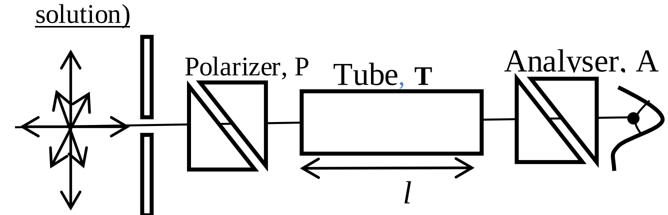
(ii) Liquid crystal display (LCD)

(iii) Reducing glare in sun glasses i.e. reducing the intensity of the incident sun light

(iv) Holography to produce an illusion of a 3D view of a scene by projecting 2 overlapping pictures of slightly different views on the same screen.

(v) In stress analysis of glass, Perspex, polythene, plastics under stress

1. <u>**Saccharimetry** (measuring concentration of sugar solution)</u>




The sacharimeterconsists of a tube T between two nickel prisms called polarizer and analyzer respectively.
The polarizer produces planepolarized light from the source and the analyzer is used to observe the emerging light.

Before putting in the solution in the tube, the analyzer is rotateduntil light from P is just cut-off. The tube T is then filled with the solution and on looking through A, light will be seen.

The analyzer A is then rotated until the light is again just cut-off from P. the angle of rotation è is measured. And the concentration of the solution is calculated from the expression;
è $= \boldsymbol{SCL}$

Where, S→ Specific rotation /Rotary power
L→Path length
C→Concetrationofthesolutionundertest

Optical Activity is the ability of a solution to rotate the planes of vibration of light passing through it.

**2. Stress analysis**

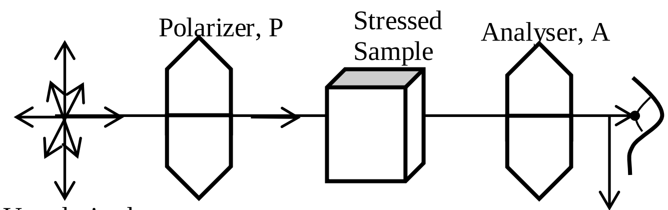
(Photo elasticity or Photo elastic stress analysis)

When glass, Perspex, polythene & some plastics are under stress they become doubly refracting.

Photo elasticity is the ability of amaterial under stress to doubly refract light incident on them producing bright and dark fringes around the area of stress.

If a sample of stressed glass is viewed in white light between two crossed, Polaroids, a pattern of coloured interference fringes are seen round the regions of high stress.

The pattern of fringes varies with the stress and its study gives information on changes in stress distribution.




The device above is called a strain viewer;it's used to detect regions of high strains.

The observed interference pattern is a result of interference between the E-rays and O-rays produced by double refraction in the stressed sample.

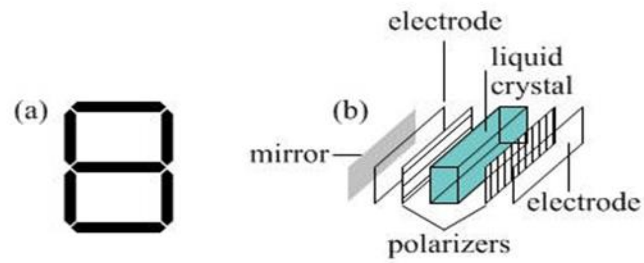
**3. Liquid crystal Displays (L.C.D)**

The term liquid crystal is used to describe a substance in astate between liquid and solid but which exhibits the propertiesof both. Molecules in liquid crystals tend to arrangethemselves until they all point in the same specific direction.

This arrangement of molecules enables the medium to flow asa liquid.

LCDs are observed in digital watches and calculators, monitors, and LCD T.Vs. An LCD read out is un readable if seen through polarized sun glasses. This is because, light coming from the L.C.D is polarized and thus can all be absorbed by polarizing sun glasses when the display is at a particular angle.

Key components are thecrossed polarizers, separated by liquidcrystals, and the mirror surface at the back.




Light incident on the display from the rightfirst passes through one polarizer, then through layers of liquid crystals. Successive layers ofliquid crystals are rotated with respect to oneanother, and the net effect is that thepolarization direction of the light is rotated by90°.

This aligns the light so that it passesthrough a second polarizer, with it transmission axis perpendicular to the first. Thelight then reflects off the mirror and reversesthe steps, emerging from the sandwich.

# MAGNETISM

**Magnet**
A magnet is a substance which has the capacity of attracting and holding the other substance e.g iron, steel, Nickel etc.
It always points in north and south directions if it is freely suspended.

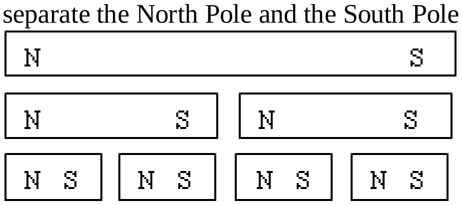
**Magnetic Materials**
Substances that can be attracted by magnet are called magnetic materials.

<u>Ferromagnetic materials</u>
Ferromagnetic materials are materials that are strongly attracted by the magnet e.g steel and iron.

<u>Non-Ferromagnetic materials</u>
Ferromagnetic materials are materials that are very weakly affected by the magnet. They are of two categories; Diamagnetic and Paramagnetic materials.
(i) Paramagnetic materials are materials that are slightly attracted by a strong magnetic field e.g Wood, Alluminium, brass, copper, platinum etc.
(i) Diamagnetic materials are materials that are slightly repelled by a strong magnetic field e.g Zinc, Bismuth, sodium chloride, gold, mercury, e.t.c.
Diamagnetic materials become weakly magnetized in a direction opposite to the magnetizing field.

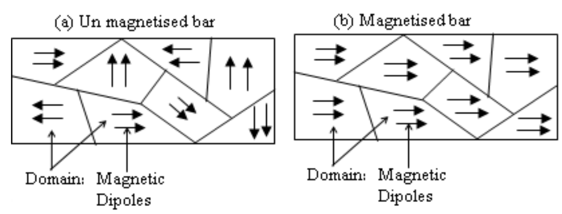
**Non- magnetic Materials**
Materials that can‘t be attracted by a magnet e.g wood, copper, rubber etc are called non-magnetic materials.

**The domain theory**
A magnet is made up of small magnets lined up with their north poles pointing in the same direction, this is illustrated when the magnet is broken into two pieces intending two separate the North Pole and the South Pole as shown below.




No matter how many times the magnet is broken, the small pieces will steel be magnets. These atomic magnets are called magnetic di- pots.
In a magnetic substance e.g magnetized steel bar, there are a number of magnetized region called domains. Domains are a grip of atoms which are tied magnets called dipoles.

Domain theory states that in an un magnetized substance the dip holes in all domains are not aligned, so when the magnet is made, the domain are aligned in the same direction
Once they are all aligned, the substance can’t be magnetized any further and it is said to be magnetically saturated.



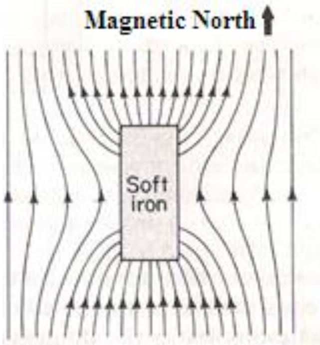

**QN**. <u>Explain in terms of the molecular theory how a steel bar gets magnetized and demagnetized.</u>
When a magnet is stroked on the steel bar the magnet domain are forced to align in the direction of the magnetic field from the magnet. They do so and remain in that direction hence the bar gets magnetized.
However, when a magnet is heated strongly, dropped on a rough surface or alternating current passed through it, the domain is set to point in opposite directions which aren’t north – south hence weakening the magnet. This is called demagnetisation

**Magnetic saturation:**
Is the limit beyond which the strength of a magnet can’t be increased at constant temperature.

**QN.** <u>Explain why increase in temperature destroys the magnetism of a magnet.</u>
When a substance is heated, molecules vibrate with greater energy, these increased vibrations destroy alignment of tiny magnets in the domain and the magnetism is decrease.

**Magnetic shielding or screening**

This is the creation of a magnetically neutral space or region in the neighbourhood of the magnetic field irrespective of the strength of the field.




All lines of force incident on the ring induce magnetism into it. These create a neutral region inside the ring
Magne08tic shielding can be applied
i) In non digital watches

ii) In T.V tubes and cathode ray tubes

iii) In electron beams

They are used to shield them from external magnetic field by placing a strong iron cylinder along the neck of the tube.

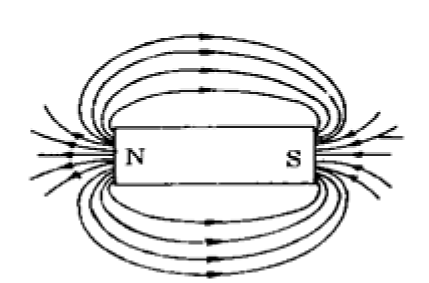
**MAGNETIC FIELDS**
A magnetic Field is a region or space in which:
a) A magnetic dipole (magnet) experiences a force.

b) A current carrying conductor experiences a force or a moving charge experiences a force

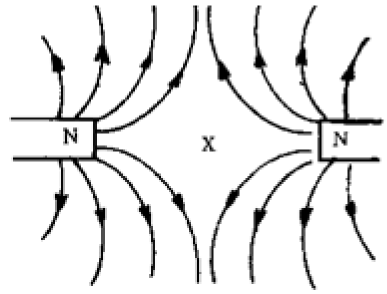
c) an emf is induced in a moving conductor

Field lines are used to represent the direction and magnitude of the magnetic field. The strength of the magnetic field is proportional to the density of the field lines.
The direction o the magnetic field is represented by the magnetic field lines. The magnetic field lines are taken to pass through the magnet, emerging from the North Pole and returning via the South Pole. The lines are continuous and do not cross each other.

Magnetic field pattern of;
a) Isolated bar magnet



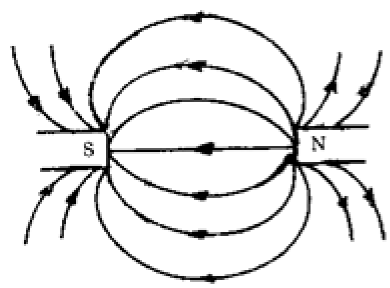

b) Two similar poles close together.

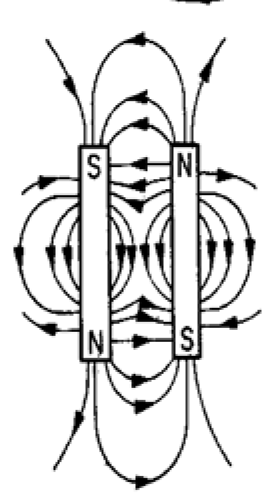



X = Nuetral Point

N.B **Neutral point-** is a point in the magnetic field, where the resultant magnetic field is zero. (i.e field lines cancel each other).

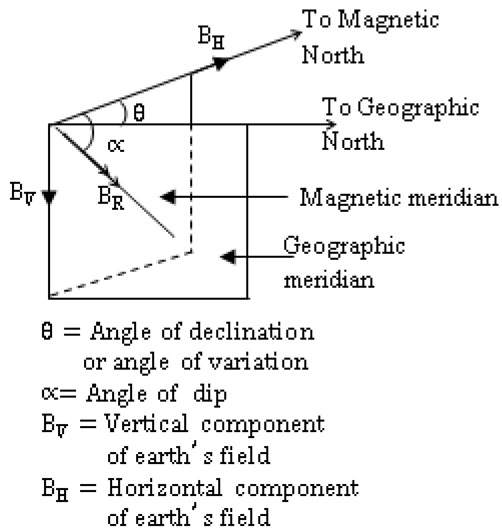
c) Two different poles near each other




**EARTH AS A MAGNET**
A freely suspended magnetic needle points towards the north and south. This shows that the earth it's self is huge magnet.

The south pole of which is some where near the geographic pole.




**Magnetic meridian:** this is the vertical plane containing or passing through the earth's magnetic north and south poles

**Geographical meridian:** This is the vertical plane passing through the geographical north and south directions

**Angle of Dip, á:** This is the angle between the earth's magnetic field and the horizontal; **OR** Angle of dip is the angle that the axis of a freely suspended bar magnet makes with the horizontal when the magnet sets.

**Angle of declination (Magnetic variation,□)** is the angle between the earth's magnetic and geographical meridian
This is the angle between geographic North Pole and the magnetic north pole.

**Magnetic axis:** is the imaginary line passing through the earth's magnetic north and south poles

**Geographical axis:** This is the imaginary line through the center of the earth and passing through the geographical north and south

**Variation of Angle of dip, áas one moves from the magnetic equator up to the North Pole**

**Definition:**
**Magnetic Equator:** This is the greatest circle in a horizontal plane perpendicular to the magnetic meridian where a freely suspended bar magnet experiences zero magnetic dip

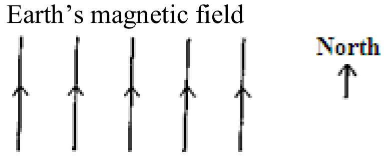
**Explanation**
At the magnetic equator, the earth's magnetic field lines are parallel to the horizontal; therefore the angle of dip at the equator is zero,$(á = 0^0)$

As one moves along a given longitude towards the North Pole, the resultant magnetic field lines meet the earth's surface at angles greater than $0^0$ but less than $90^0$ thus the angle of dip at such a position is also greater than zero but less than $90^{0.}$
i.e. $0^0 < á < 90^0$
At the North Pole, the magnetic field lines are normal to the surface of the earth, thus they are perpendicular to the horizontal. Therefore the angle of dip at the North Pole equals $90^0$ i.e. , $á = 90^0$.

Generally, the angle of dip increases from $0^0$ at the equator up to $90^0$ at the North Pole

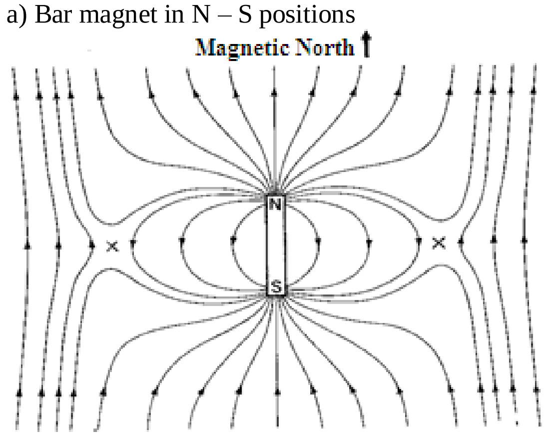
**Earth's magnetic field;**
This is the series of parallel lines running rom geographic south to geographic north as shown below.

Earth's magnetic field

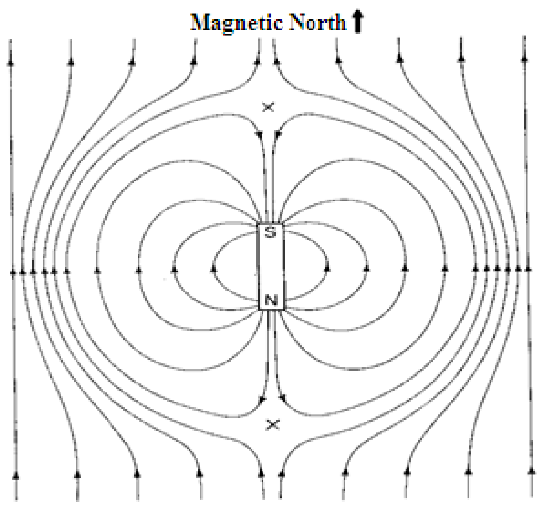



**Interaction of earth's field with a bar magnet**.

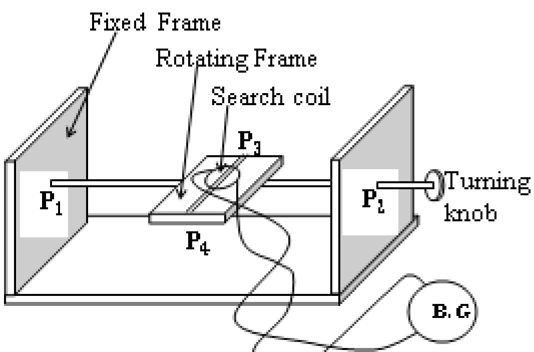
a) Bar magnet in N – S positions




b) Bar magnet in S – N positions




**Measurement of the Earth's magnetic field**
**(a) Using an earth inductor**
An earth inductor is a device used for measuring the horizontal, $B_H$ and vertical, $B_V$ components of the earth's magnetic fields.
Hence the Earth's resultant magnetic field, B and angle of dip, $\propto$ can be determined.




(i) <u>Measurement of the earth's horizontal magnetic component, $B_H$</u>.
A search coil of known number of turns, N and cross-section area A, is connected in series with a ballistic (B.G) such that the total resistance, R of the B.G and the search coil are known.
The plane of the coil is made to coincide with that of the rotating frame.
The planes of the coil and that of the rotating frame are made vertical in such a way that they are perpendicular to the earth's magnetic meridian with the help of a compass needle. The plane of the coil is now threaded normally by the earth's horizontal magnetic component, $B_H$.
The rotating frame is then quickly rotated through $180^0$ a bout the horizontal axis and the maximum deflection, $\theta_H$ on the ballistic galvanometer is noted.
The earth's horizontal magnetic component $B_H$ is then calculated from;

$$\mathbf{B_H} = \frac{\mathbf{kR\theta_H}}{\mathbf{2NA}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i).$$

Where k is the sensitivity torsion constant of the ballistic galvanometer (B.G) obtained by charging a standard capacitor, $C_s$ to a known P.d, $V_s$ and then discharging it through the B.G. The maximum deflection $\theta_s$ is read and noted.

$$\mathbf{k} = \frac{C_sV_s}{\theta_s}.$$

(ii) <u>Measurement of the earth's horizontal magnetic component, $B_H$</u>.
A search coil of known number of turns, N and cross-section area A, is connected in series with a ballistic (B.G) such that the total resistance, R of the B.G and the search coil are known.
The plane of the coil is made to coincide with that of the rotating frame.
The planes of the coil and that of the rotating frame are made horizontal in such a way that they are perpendicular to the earth's magnetic meridian with the help of a compass needle.
The plane of the coil is now threaded normally by the earth's vertical magnetic component, $B_V$.
The rotating frame is then quickly rotated through $180^0$ a bout the vertical axis and the maximum deflection, $\theta_V$ on the ballistic galvanometer is noted.
The earth's vertical magnetic component $B_V$ is then calculated from;

$$\mathbf{B_V} = \frac{\mathbf{kR\theta_V}}{\mathbf{2NA}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii).$$

The earth's resultant magnetic flux density, $\mathbf{B_R}$is then calculated from;

$$\mathbf{B_R} = \sqrt{(\mathbf{B_H})^2 + (\mathbf{B_V})^2}$$

(iii) <u>Measurement of the angle of dip, $\propto$</u>
From equations (i) and (ii);

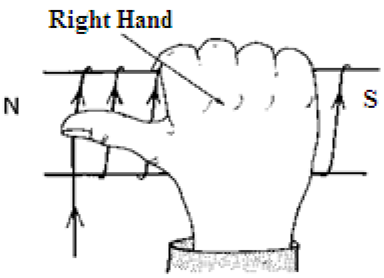
$$\mathbf{tan} \propto = \frac{\mathbf{B_V}}{\mathbf{B_H}}$$

$$\propto = \mathbf{tan}^{-1}\left(\frac{\grave{e}_{\mathbf{V}}}{\grave{e}_{\mathbf{H}}}\right)$$

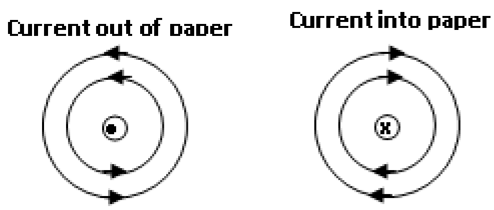
**Magnetic effect of an electric current.**
Any straight conductor carrying current experiences a magnetic field around it. The direction of a magnetic field around the conductor is given by the right hand grip rule. which states that imagine a conductor to be griped in the right hand with the thumb pointing in the direction of the magnetic field, then the fingers will point in the direction of the current.

**Right hand grip rule**
It states that imagine a conductor to be griped in the right hand with the thumb pointing in the direction of the magnetic field, then the fingers will point in the direction of the current.

Grip the soft iron bar with the right hand figure, following the direction of current. The end where the thumb points is the north pole.

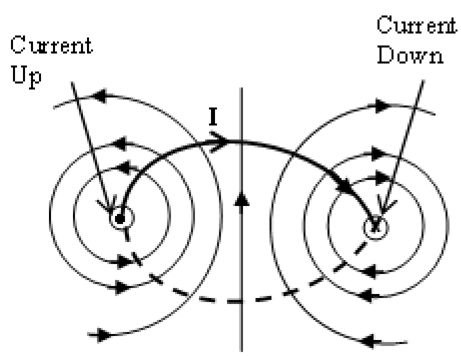



**(i) Magnetic fields due to a straight wire carrying current**



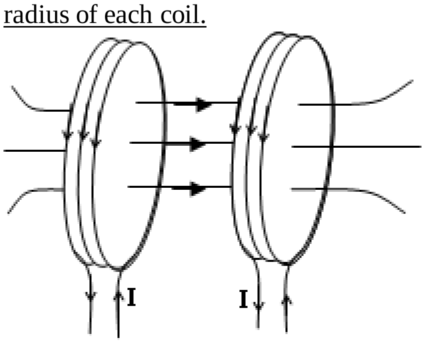

**Maxwell's right hand rule:**
This is used to find the direction of the field.
If one grasps the current carrying straight wire in the right hand with the thumb pointing in the direction of current, then the fingers curl pointing in the direction of the magnetic field.

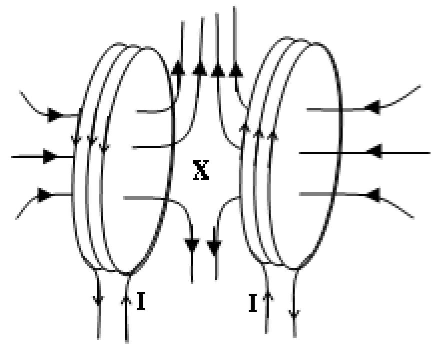
**(ii) Magnetic field due to a current carrying circular coil.**



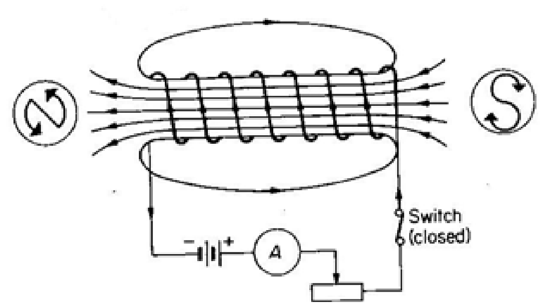

Magnetic fields near the center of the circular coil are uniform hence the magnetic field lines are nearly straight and parallel.

<u>Magnetic field patterns which results when two identical circular coils are placed coaxially at a distance equal to the radius of each coil.</u>



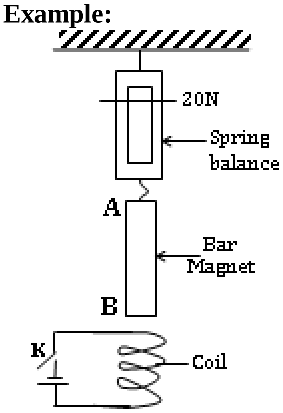




**(ii) Magnetic fields due to a solenoid**
A solenoid can be viewed as consisting o many circular coils, wound very closely to each other.



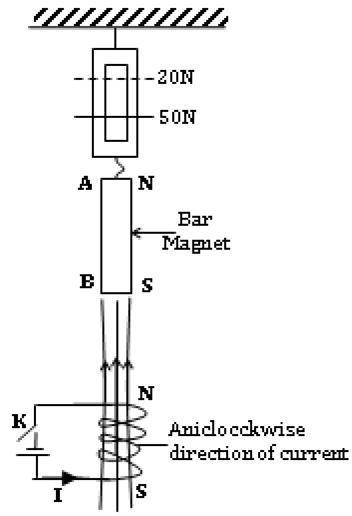

Use right hand grip to determine direction of field. (Fingers show direction of current; thumb direction)
In the middle of the solenoid, the magnetic field lines are parallel to the axis of the solenoid. At the ends of the solenoid, the lines diverge from the axis. The magnetic field due to a current carrying solenoid resembles that of a bar magnet. The polarities of the field are identified by looking at the ends of the solenoid. If current flow is clockwise, the end of the solenoid is South Pole, and if anticlockwise then it is North Pole.

**Example:**



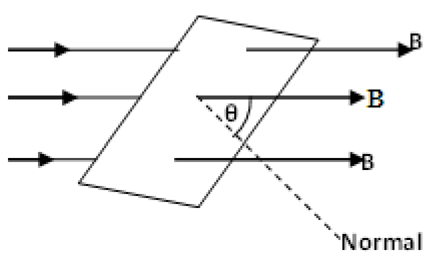

The diagram above shows a piece of soft iron bar suspended freely on a spring balance. One end of the bar is close to the end of a coil connected to a source of e.m.f via switch K. When switch K is closed, the spring balance reads 50N. State the polarity of magnet AB and explain your answer.

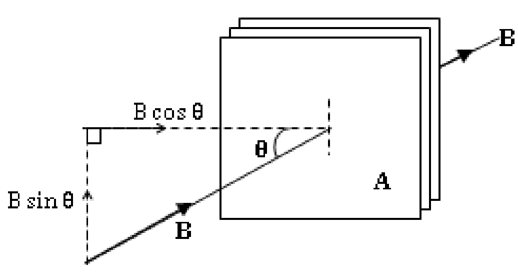
**Solution:**




The upper end of the coil is a north pole when K is closed.
Due to the increase in the spring balance reading, (i.e from 20N to 50N), it implies that the magnet is attracted towards the coil. Hence end A is a south pole and B is a north pole.
Looking from the top of the coil, current flows in the anti-clockwise direction.
Thus the top of the coil becomes the North Pole since the magnet is attracted. This implies that side A is a south pole hence increasing the spring balance reading.

**MAGNETIC FLUX**
Magnetic flux is a measure of the number of magnetic field lines passing normally across a given area of space.
Consider an area A, the normal of which makes an angle, è with the uniform magnetic field of flux density B.




The component of B along the normal to the area is$\mathrm{B}\cos\theta$.

The product $\phi = A(\mathrm{B}\cos\theta)$ is called the magnetic flux through the area.

**Magnetic flux** is the product of the area of projection normal to the magnetic field lines and the magnetic flux density along the normal to the area.
If the magnetic field is perpendicular to the area A, then magnetic flux,$\boldsymbol{\phi} = \mathbf{A}(\mathbf{B}\,\mathbf{cos}$ è$)$.
But, $\theta = 0$; Thus; $\phi = \mathrm{AB}$

For a coil of N turns, the total flux linking the coil is called the <u>magnetic flux linkage</u>, **Φ**.
Magnetic flux linkage,Ö $= \mathbf{N\phi} = \mathbf{NAB\,cos}$ è.

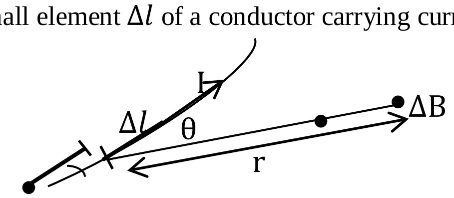
The unit of magnetic flux is the **webber (Wb).**

Therefore from $\phi = \mathrm{AB}, \Leftrightarrow \mathrm{B} = \dfrac{\phi}{\mathrm{A}}$

**Magnetic flux density (or magnetic induction)** is the magnetic flux threading an area of $1\mathrm{m}^2$, placed with its plane perpendicular to the magnetic field.
Magnetic flux density is the force acting on a straight conductor of length one metre, carrying a current of one ampere at right angles to the direction of the field.

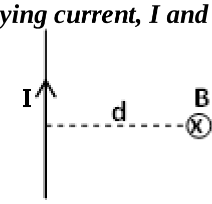
The unit of magnetic flux density is a **tesla**(**T**).
A tesla is the magnetic flux density of a straight conductor of length **1m** placed across the field and carrying a current of **1A** such that it experiences a magnetic force of **1N.**

**Biot-Savart's law**
For any shape of a conductor, the flux density, $\Delta\mathrm{B}$, due to a small element $\Delta l$ of a conductor carrying current is given by;

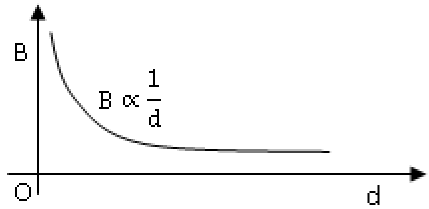



$$\Delta\mathrm{B} \propto \frac{\mathrm{I}\Delta l\sin\theta}{\mathrm{r}^2} \Leftrightarrow \Delta\mathrm{B} = \left(\frac{\mu_0}{4\pi}\right)\frac{\mathrm{I}\Delta l\sin\theta}{\mathrm{r}^2}$$

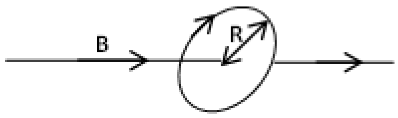
***(i) Magnetic flux density at a point a distance d from a long straight wire carrying current, I and placed in a vacuum***




$\mathbf{B} = \dfrac{\text{ì}_{\mathbf{o}}\mathbf{I}}{2\text{ð}\mathbf{d}}$
Where $\mu_o$ is the permeability in free space, $\mu_o = 4\pi \times 10^{-7}\mathrm{Hm}^{-1}$ .



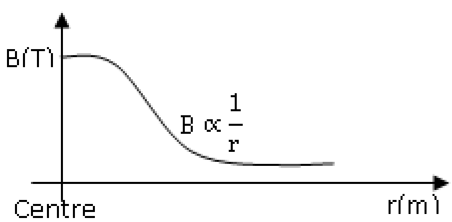

***(ii) Magnetic flux density at the center of a circular path of N turns each of turns each of radius R and carrying a current I.***




The length, $l$ of the wire in a circular coil of N turns is given by;

$$\boldsymbol{l} = \mathbf{N}(2\pi\mathbf{r})$$

$$\mathbf{B} = \frac{\mu_o \mathbf{NI}}{2\mathbf{R}}$$




***(iii) Magnetic flux density along the axis of a long solenoid of n turns per meter, each carrying a current I.***

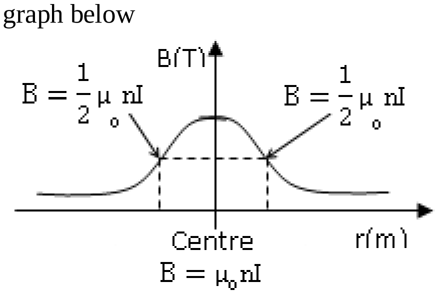
$$\mathbf{B} = \frac{\mu_o \mathbf{NI}}{\mathbf{L}} = \mu_o \mathbf{nI}$$

Where N is the number of turns and $l$ is the length of the solenoid.

At the ends of the solenoid, the magnetic flux density, B is half the value of B at the centre.

$$B_{\text{At Ends}} = \frac{\mu_o NI}{2L} = \frac{1}{2}\mu_o nI$$

The magnetic flux density goes on reducing with increase in the distance from the centre of the solenoid as shown in the graph below




$As\ x \to \infty, B \to 0$

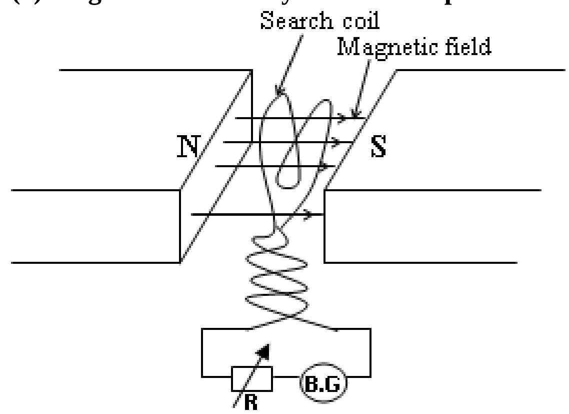
Note: If the solenoid is wound round a soft iron, the material becomes a core solenoid. Its permeability increases considerably.

Relative Permeability $\mu_r$ is the number of times the permeability of a medium, $\mu$ increases relative to that of free space or air, $\mu_0$.

$$\mu_r = \frac{\mu}{\mu_0} \Leftrightarrow \mu = \mu_r \mu_0$$

**Measurement of magnetic flux density, B**

**(a) Magnetic flux density between the poles of a magnet**




**Procedure:**

A search coil of known geometry (known number of turns, N and cross-section area A) is connected in series with calibrated ballistic galvanometer (B.G) and variable resistor, R.

The search coil is then placed in the magnetic field between the poles of the magnet in such a way that its plane is threaded normally by the magnetic field lines.

The plane of the coil is then flipped or turned equally through $90^0$ about the horizontal axis through its center. The search coil is then flipped or pulled quickly vertically down wards out of the field. The magnetic flux linking the coil changes and an e.m.f is induced in the coil. Hence a current I flows through the coil which causes the B.G to deflect.

The maximum deflection $\theta$ on the ballistic galvanometer scale is noted.

The rheostat, R is adjusted where necessary so that several values of the deflection $\theta$ can be obtained. Therefore, induced charge;

$$Q = k\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

The induced charge is proportional to the change in the magnetic field flux, $\Phi$.

$$Q = \frac{-Nd\Phi}{R} = \frac{-N(0-\Phi)}{R} = \frac{N\Phi}{R}$$

$$Q = \frac{N\Phi}{R} = \frac{NBA}{R} \ldots\ldots\ldots\ldots\ldots (ii)$$

Thus from equations (i) and (ii)

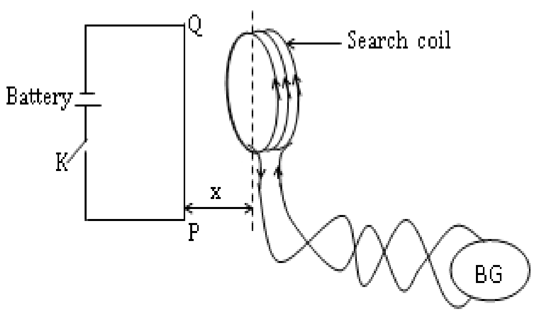
$$Q = \frac{NBA}{R} = k\theta$$

The magnetic flux density between the poles B is;

$$\mathbf{B = \frac{k\grave{e}R}{NA}}$$

Where;
R=Total resistance of the coil and the B.G
N= number of turns the of the search coil
A= Area of the plane of the coil
K= Torsion constant of the ballistic galvanometer in coulombs per radian ($Crad^{-1}$).

**(b) Magnetic flux due to a straight wire carrying current, I.**




**Procedure**

A search coil of known geometry (i.e. known number of turns, N and area, A) is connected in series with a calibrated ballistic galvanometer (B.G).
The search coil is then placed near the wire PQ such that its plane is parallel to the wire PQ
Switch K is then closed, and the maximum deflection $\theta_1$on the Ballistic Galvanometer is noted. Switch K is now opened and the maximum deflection$\theta_2$is noted.
The average deflection, $\theta$ is then determined.

$$\theta = \frac{\theta_1 + \theta_2}{2}$$

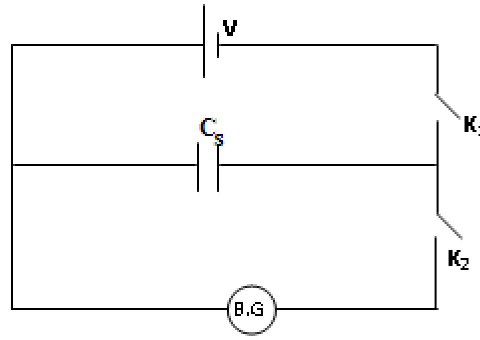
Included charge, $Q \propto \theta$

$$Q = \frac{NBA}{R} = K\theta$$

$$B = \frac{k\theta R}{NA}$$

Where R is the total resistance of B.G and the coil,
N is the number of turns of the search coil
A = Area of the plane of the coil and,
K = Torsion constant of the B.G expressed in coulombs per Radian ($Crad^{-1}$)

**Note:** If k is not known, a standard capacitor of capacitance, $C_S$ Is charged and discharged through the ballistic galvanometer by closing $k_1$ when $k_2$ is open;




$k_1$ is then opened $k_2$ is closed; the maximum deflection, $\theta_s$ is noted.

$$Q = C_sV \ldots\ldots\ldots\ldots\ldots\ldots(i)$$

$$Q = k\theta_s \ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

Thus from (i) and (ii);

$$k\theta_s = C_sV$$

$$k = \frac{C_sV}{\theta_s} \ldots\ldots\ldots\ldots\ldots\ldots(iii)$$

$$\mathbf{B = \left(\frac{C_sV}{\grave{e}_s}\right)\left(\frac{R\grave{e}}{NA}\right) = \frac{RC_sV}{NA}\left(\frac{\grave{e}}{\grave{e}_s}\right)}$$

**(c) Experiment to investigate the dependence of magnetic flux density, B at the centre of a circular coil on the current, I through the coil.**

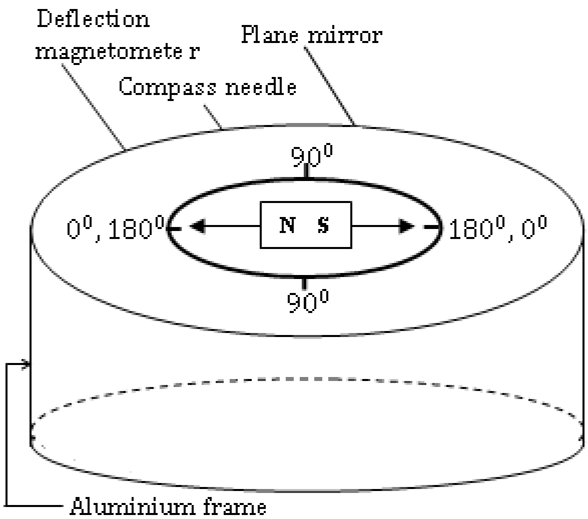
(i). Using a tangent galvanometer

(ii). Using a search coil

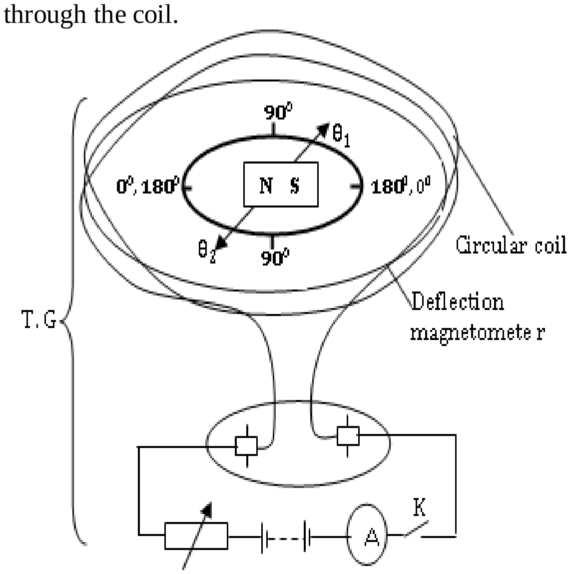
(iii). Using a current (or ampere) balance

**(i) Using a tangent galvanometer.**

A tangent galvanometer (T.G) is used for comparing two magnetic fields perpendicular to each other, one of them being the horizontal component of the earth's magnetic field, $B_H$.
The T.G consists of a deflection magnetometer mounted at the centre of the coil along its vertical diameter.
The deflection magnetometer consists of a small compass needle having two light aluminium pointers attached to it and pivoted freely inside the aluminium frame so that it swings freely about the horizontal plane about the angular scale.




**Procedure**
A coil having a deflection magnetometer at its centre is placed in the earth's magnetic meridian with no current flowing through the coil.

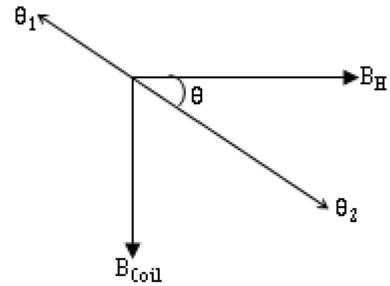



The scale of the magnetometer is then set such that the ends of the pointer are at the zero marks.
Switch K is closed to allow a suitable value of current, I to flow through the coil.
The deflections, $\theta_1$ and $\theta_2$ of the pointers are noted and the average deflection, $\theta$ calculated:

$$\theta = \frac{\theta_1 + \theta_2}{2}$$

If a reversing switch is used, the first setting of the pointer gives deflections, $\theta_1$ and $\theta_2$ while the second setting on reversing the current gives deflections, $\theta_3$ and $\theta_4$.
The average deflection $\theta$ is calculated as follows:

$$\theta = \frac{\theta_1 + \theta_2 + \theta_3 + \theta_4}{4}$$




Hence from the figure above, $B_H$ is the horizontal component of the earth's magnetic field.

$$\text{Tan}\,\theta = \frac{B_{coil}}{B_H}$$

$\Leftrightarrow B_{coil} = B_H \tan\theta \ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots$ (i)
The procedures are repeated for various values of the current, I and the results tabulated including values of $\tan\theta$.
A graph of $\tan\theta$ against I is plotted. It is a straight line graph through the origin.
$\Longrightarrow \tan\theta \propto I \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (ii)

Since $B_H$ is constant, from equations (i) and (ii);
$\Longrightarrow B_{coil} \propto I$

But the magnetic flux density at the centre of the coil is given by;

$$B_{coil} = \frac{\mu_o NI}{2r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \text{(iii)}$$

From equations (i) and (iii):

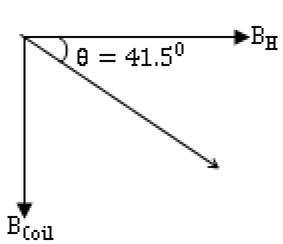
$$\mathbf{B_H = \frac{B_{coil}}{\tan\theta} = \frac{\mu_o NI}{2r\tan\theta}}$$

**Note:**
The earth's horizontal magnetic field can be determined using the same method above.

**Example:1; UNEB 2011 No. 6 (c)**
A coil of 50 turns and radius 4cm is placed with its plane in the earth's magnetic meridian. A compass needle is placed at the centre of the coil. When a current of 0.1A passes through the coil, the compass needle deflects through $40^0$. When the current is reversed, the needle deflects through $43^0$in the opposite direction. Calculate the;
(i) Horizontal component of the earth's magnetic field.
(ii) Magnetic flux density of the earth at the place where the angle of dip is$15^0$. .

**Solution:**
(i) $\theta_1 = 40^0; \theta_2 = 43^0$.
The average deflection, $\theta$ is calculated as follows:

$$\theta = \frac{\theta_1 + \theta_2}{2} = \frac{40^0 + 43^0}{2} = 41.5^0$$



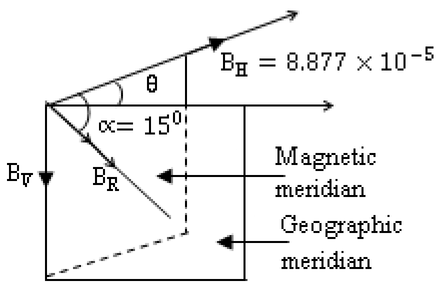

Using: $\tan\theta = \frac{B_{coil}}{B_H}$

$$B_H = \frac{B_{coil}}{\tan\theta} = \frac{\mu_o NI}{2r\tan\theta}$$

$$B_H = \frac{4\pi\times10^{-7}(50\times10)}{2\times4\times10^{-2}\tan41.5}$$

$$B_H = 8.877\times10^{-5}\ \text{T}$$

(ii)




$B_H = B_R\cos\theta \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (i)$
$B_V = B_R\sin\theta \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (ii)$

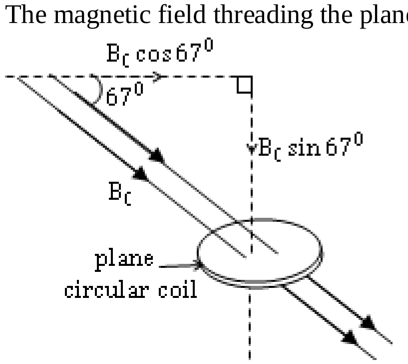
Thus from equation (i),

$$B_R = \frac{B_H}{\cos\theta}$$

$$B_R = \frac{8.877\times10^{-5}}{\cos15^0} = 9.19\times10^{-5}$$

**Example: 2**
A plane circular coil of radius 5cm and 20 turns is placed with its surface flat on a horizontal table. If the plane of the coil is threaded by a magnetic field of flux density $3.6\times10^{-5}$ T at an angle of dip of $67^0$, find the magnetic flux threading the coil.

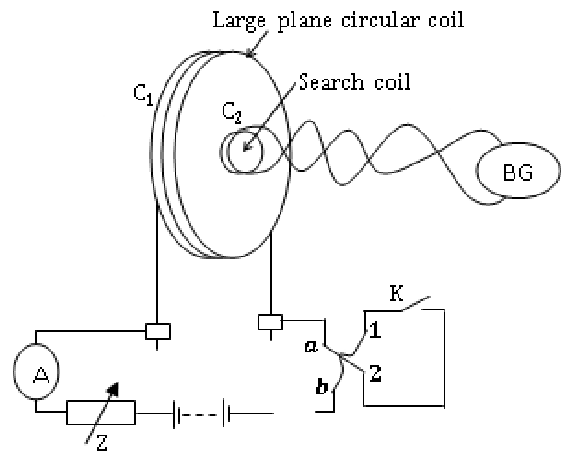
**Solution:**
The magnetic field threading the plane of the coil normally.




$\phi = AB$
$\phi = AB_C\sin67^0$
$\phi = (\pi r^2)B_C\sin67^0$
$\phi = 3.14\times(5\times10^{-2})^2\times3.6\times10^{-5}\times\sin67^0$
$\phi = 2.6\times10^{-2}\text{Wb}$

**UNEB 1998 No. 1 (a)**, **1989 No. 5 (a)**

**(ii) Using a search coil.**




**Procedures:**
A search coil of known geometry (i.e. known number of turns, N and area, A) is connected in series with a calibrated ballistic galvanometer (B.G).
The search coil is then placed at the centre of a large coil connected to the source of e.m.f via a reversing switch as shown in the diagram above.
For a given setting of the rheostat Z, switch K is closed and the reversing contact, a connected to terminal 1 while b is connected to terminal 2 of the coil. Current I flows through the coil causing a magnetic field of intensity, B at the centre of $C_2$.
The maximum deflection $\theta_1$ of the ballistic galvanometer is noted.
Keeping K closed, the reversing switch terminals are interchanged (i.e. a to 2 and b to 1).
The maximum deflection $\theta_2$ of the ballistic galvanometer in the opposite direction is noted.
The average deflection, calculated from;

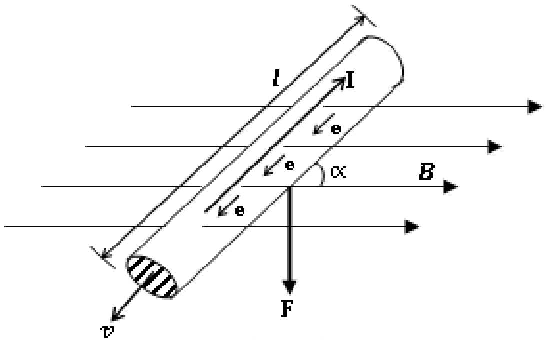
$$\theta = \frac{\theta_1+\theta_2}{2}$$

The Magnetic flux density, B at the centre of a circular coil is calculated from;

$$\mathbf{B = \frac{k\grave{e}R}{NA}}$$

**Force on a charge moving in a magnetic field.**
(<u>Deducing the force on a current carrying conductor</u>)

<u>Force of a charge in a magnetic field.</u>
Consider a straight conductor (wire) of length l, cross section area, A and carrying a current I, in a uniform magnetic field of flux density B.




The current flowing in the conductor (wire) is given by;

$\mathbf{I} = \frac{\mathbf{Q}}{\mathbf{t}} = \mathbf{ne}\boldsymbol{v}\mathbf{A}$ ………………………………….. $(i)$.

Where $\boldsymbol{n}$ is the number of electrons per unit volume,A is the cross-sectional area of the conductor, $v$ is the drift velocity of electrons, and eis the electron charge.

The magnetic force on the wire is given by;

$$\mathbf{F} = \mathbf{BI}\boldsymbol{l}\,\mathbf{sin} \propto \text{………………………………….} (ii)$$

Putting equation (i) into (ii) gives;

$\mathrm{F} = \mathrm{B(ne}v\mathrm{A)}l \sin \propto$

$\mathrm{F} = (\mathrm{nA}l)\mathrm{Be}v \sin \propto$

But,$(\mathbf{nA}\boldsymbol{l}) = \boldsymbol{N} = \text{Total number of electrons in the conductor}$. Thus;

$\mathrm{F} = \mathrm{NBe}v \sin \propto$

Therefore, magnetic force on the wire having N electrons is given by;

$\mathbf{F} = \mathbf{NBe}\boldsymbol{v}\,\mathbf{sin} \propto \cdots$ …………………….. $(iii)$

Thus the Force on one electron in the wire is;

$\mathrm{F} = \mathrm{Be}v \sin \propto$

But, $\boldsymbol{Ne} = \boldsymbol{q} = \text{Total Charge in the Conductor}$.

There fore the force, $\mathrm{F_q}$ on any particle of charge, q moving with velocity $v$ in a uniform magnetic field of flux density B is given by;

$\mathbf{F_q} = \mathbf{Bq}\boldsymbol{v}\,\mathbf{sin} \propto$.

If the particles velocity, $v$ is at right angles to the uniform magnetic field, B, then $\propto = 90^0$. Hence

$\mathrm{F_q} = \mathrm{Bq}v \sin 90^0$.

$\mathbf{F_q} = \mathbf{Bq}\boldsymbol{v}$

$$\mathrm{B} = \frac{\mathrm{F_q}}{\mathrm{q}v}$$

Hence *magnetic flux density B at a point in a magnetic field is the force exerted on a charge of 1C, moving with a velocity of 1ms$^{-1}$, at right angles to the magnetic field.*

Note: To derive $\mathbf{F} = \mathbf{BI}\boldsymbol{l}\,\mathbf{sin} \propto$, begin from the force on a single electron in a current carrying conductor ($\mathbf{F_e} = \mathbf{Be}\boldsymbol{v}\,\mathbf{sin} \propto$).

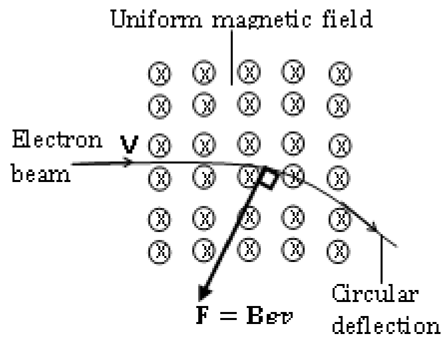
Substitute for v, from,$\mathbf{I} = \frac{\mathbf{Q}}{\mathbf{t}} = \mathbf{ne}\boldsymbol{v}\mathbf{A}$

Then total force on the conductor; $\mathbf{F} = \mathbf{NF_e}$

Where $\mathbf{N} = (\mathbf{nA}\boldsymbol{l})$

Motion of a charged particle in a uniform magnetic field

Suppose a positively charged particle is projected with velocity $v$ at right angles to a uniform magnetic field o flux density B.




The force on the particle, $\mathrm{F} = \mathrm{Bq}v$ is at right angles to $v$ (using Fleming's left hand rule.)

The rate of change of Kinetic energy (K.E) of the particle is given by;

$$\frac{\mathrm{d(K.E)}}{\mathrm{dt}} = \mathrm{F}.v = 0(\text{since F is perpendicular to } v$$

Hence Kinetic energy, $K.E = \frac{1}{2}mv^2 = constant.$

It follows that the speed of the charged particle remains constant. Hence the particle moves in a circular path of radius r.

<u>For circular motion in a uniform magnetic field;</u>

$\text{Magnetic force} = \text{Centipetal force}.$

$$\mathrm{Bq}v = \frac{mv^2}{r}$$

$$\mathrm{r} = \frac{mv}{Bq}$$

$$\text{Period time, T} = \frac{2\pi}{\omega} = \frac{2\pi \mathrm{r}}{\mathrm{v}} = \frac{2\pi \mathrm{m}}{\mathrm{Bq}}$$

$$\text{Frequency, f} = \frac{1}{\mathrm{T}} = \frac{\mathrm{Bq}}{2\pi \mathrm{m}}$$

The kinetic energy of the particle is given by

$$\text{Kinetic energy}, K.E = \frac{1}{2}mv^2$$

$$\text{But v} = \frac{Bqr}{m}$$

$$\text{Kinetic energy}, K.E = \frac{1}{2}m\left(\frac{Bqr}{m}\right)^2$$

Kinetic energy, $\mathbf{K.E} = \dfrac{(\mathbf{Bqr})^2}{2\mathbf{m}}$

**Example:**

An electron beam moving with a velocity of $10^6 ms^{-1}$ moves through a uniform magnetic of flux density 0.1T which is perpendicular to the direction of the beam.

(a) Draw a sketch to show the direction of the beam, field and force.

(b) Calculate the:

(i) Force on the electron, if the electron charge is $-1.6 \times 10^{-19}$C.

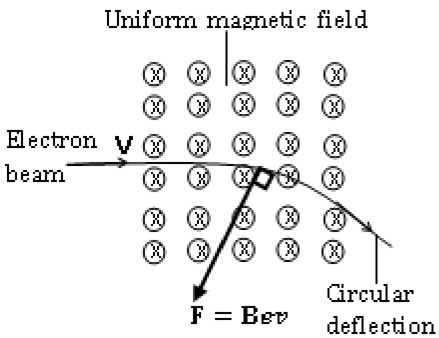
(ii) Period.

(iii) Radius of the path described by the beam.

$[\text{Mass of an electron}, m_e = 9.1 \times 10^{-31}\text{kg}]$

**Solution:**

(a)




(b) (i)

$\text{F} = \text{Be}v$

$\text{F} = 0.1(1.6 \times 10^{-19})(10^6)$

$\text{F} = 1.6 \times 10^{-14}\text{N}$

(ii)

$$\text{Period time, T} = \frac{2\pi m}{Be}$$

$$T = \frac{2\pi(9.1 \times 10^{-31})}{0.1(1.6 \times 10^{-19})}$$

$$T = 3.57 \times 10^{-10}\text{s}$$

(iii)

Radius of the path described by the beam;

Magnetic force = Centipetal force.

$$\text{Be}v = \frac{mv^2}{r}$$

$$r = \frac{mv}{Be}$$

$$r = \frac{(9.1 \times 10^{-31})(10^6)}{0.1(1.6 \times 10^{-19})}$$

$$r = 5.69 \times 10^{-15}\text{ m}$$

**Exercise:**

1. An alpha particle travels in a circular path of radius 0.5m in a magnetic field of flux density 1.0T. Calculate the;

(i) Period$[T = 6.65 \times 10^{-27}]$

(ii) Speed $[v = 2.41 \times 10^7 \text{ms}^{-1}]$

(iii) Kinetic energy of the alpha particle.

$[\text{K.E} = 1.93 \times 10^{12}\text{ J}]$

$[\text{Mass of an alpha particle} = 6.65 \times 10^{-27}\text{kg}]$

2. An ion beam consisting of $^{24}Mg$ and $^{26}Mg$ ions (Masses 24.0u and 26.0u respectively) is accelerated through a p.d of $2 \times 10^3$V. The beam then enters a region of uniform magnetic field of flux density $5.0 \times 10^{-2}$ T. Calculate difference in the radii of the orbits of the ions in the magnetic field.

$[\text{Difference in radii} = 60.03\text{m}]$

**THE HALL EFFECT**

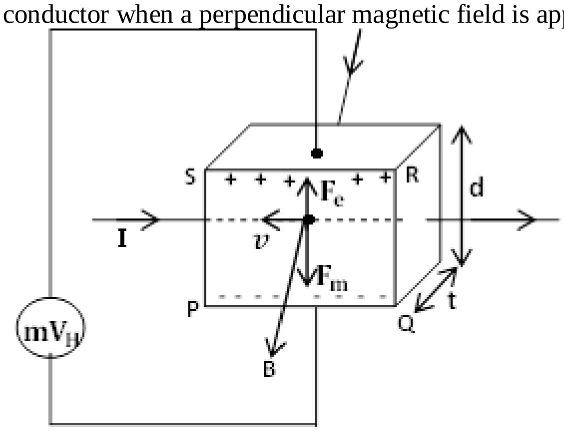
A current carrying conductor in a magnetic field has a small potential difference across its sides at right angles to the field.

Hall Effect:

This is the process where an e.m.f is set up across a current carrying conductor when a perpendicular magnetic field is applied.

Hall voltage:$\mathbf{V_H}$

This the maximum p.d set up across a current carrying conductor when a perpendicular magnetic field is applied.




Consider a rectangular metal slab of thickness, t width, d and length, *l* carrying current, I in a uniform magnetic field of flux density, B

The orientation is such that B is normal to the longest face, PQRS and the current, I is passed through the opposite face (smallest face).

The flow of electrons is in opposite direction to a conventional current flow. If the metal is placed in a uniform magnetic field B, at right angles to the face PQRS, of the slab and directed out of paper.

The electrons moving with an average drift velocity, v experience a magnetic force, $\mathbf{F_m} = \mathbf{Be}\boldsymbol{v}$ which acts on each electron in the direction from RS to PQ. (Fleming's left hand rule).

Therefore electrons accumulate along the side PQ of the metal making it negatively charged and RS positively charged. Large opposite charges are thus set up at the opposite faces.
Hence a p.d or emf which opposes the electron flow is set up across the slab.

This effect is called the hall effect (i.e. process of producing this emf).
The flow of electrons across the slab (separation of the charges) continues until no more charge is able to move from face SR to PQ. The flow of electrons ceases when the emf reaches a particular maximum value called hall voltage.

The magnitude of Hall Voltage
An electric field of intensity, E is set up between the charged faces causing an electric force, $\mathbf{F_e} = \mathbf{Ee}$
This force which is directed upwards from PQ to RS is equal to the force produced by the magnetic field, when the electrons are in equilibrium. i.e. when the electrons are not deflected. Thus;

At equilibrium, when Hall voltage is attained;

$$\text{Electric foorce} = \text{Magnetic force}$$
$$F_e = F_m$$

$$Ee = Bev$$
$$E = Bv$$

$$\text{Where; } E = \frac{V_H}{d}, \text{and } v = \frac{I}{neA}$$

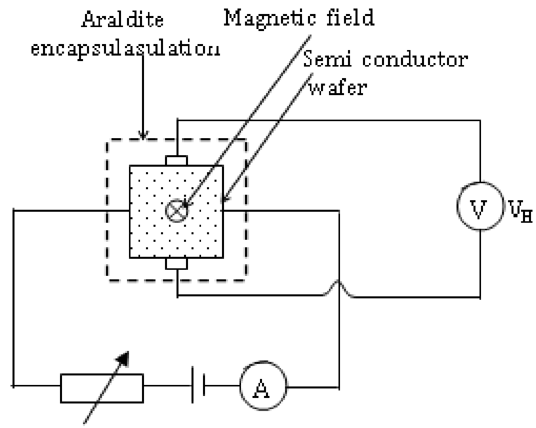
$$\frac{V_H}{d} = B\left(\frac{I}{neA}\right)$$

$$\text{But, Cross sectional area, } A = t \times d$$

$$\frac{V_H}{d} = B\left[\frac{I}{ne(td)}\right]$$

$$V_H = \frac{BI}{net}$$

**Determination of the magnetic flux density, B of a field using the Hall Effect**
The Hall Effect provides a practical method of measuring magnetic flux density.




The thickness, t of the conductor is determined. Let n be the number of electrons of charge, e per unit volume, that drift through the conductor.
Current, I is passed across the opposite sides of the semi-conductor wafer until the e.m.f becomes steady..
The voltmeter reading, $V_H$ is determined using a high impedance voltmeter.
The experiment is repeated for different values of I and the corresponding values of $V_H$ obtained.
A graph of $V_H$ against I is plotted and its slope, S determined.
The magnetic flux density, B is then obtained from the expression;

$$\mathbf{B = net}\boldsymbol{S}$$

Alternatively.
A given strip (hall probe) is calibrated by measuring the hall voltage $V_0$ to a given current in a magnetic field of known magnetic flux density, $B_0$. The magnetic flux density B of the unknown magnetic field can then be determined by placing the strip in the magnetic field and measuring the hall voltage, V.

$$V_0 = \frac{B_0 I}{net} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

$$V = \frac{BI}{net} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

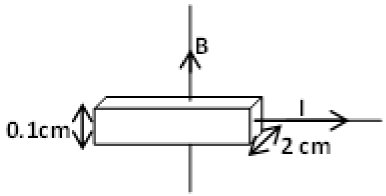
Equation (ii) divides by equation (i)

$$\frac{V}{V_0} = \frac{B}{B_0} \Leftrightarrow V = \left(\frac{B}{B_0}\right)V_0$$

**Exercise**
1. A metallic strip of width 2.5cm and thickness 0.1cm carries a current of 10amps. When a magnetic is applied normally to the broad side of the strip, a hall voltage of 2mV develops. Find the magnetic flux density if the conduction electron density is $6.0\times10^{28}m^{-3}$.

2. A metal strip 2cm wide and 0.1cm tick carries a current of 20A at right angles to a uniform magnetic field of flux density 2T. The hall voltage is 4.27mV.

**Solution:**



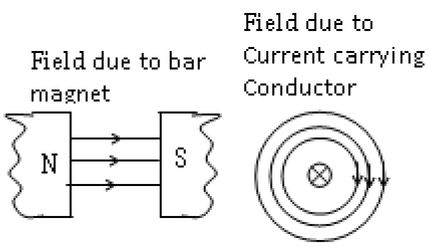

Calculate the;
(i) Drift velocity of the electrons in the strip
(ii) Density of charge carriers in the strip.

(i)
$V_H = Bvd$

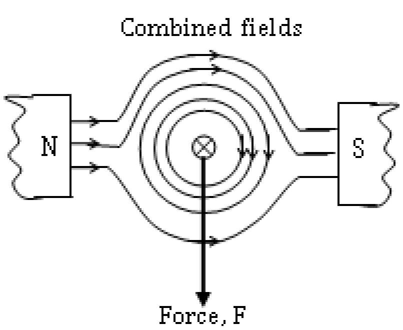
$$v = \frac{V_H}{Bd} = \frac{4.27 \times 10^{-6}}{2(2 \times 10^{-2})} = 1.067 \times 10^{-4} ms^{-1}$$
(ii)
$$j = \frac{I}{A} = nev$$

$$n = \frac{I}{evA}$$
$$n = \frac{20}{(1.6 \times 10^{-19})(1.067 \times 10^{-4})(0.2 \times 10^{-4})}$$
$$n = 5.84 \times 10^{28} m^3$$

**Force on a current carrying conductor in a magnetic field**
(a) Origin of the force that causes motion of a current carrying conductor placed across a magnetic field.
When a current carrying conductor is placed across a magnetic field, it sets up a magnetic field around itself.
The two fields then interact with each other causing a resultant force.




The combined field exert a force on the current carrying conductor. The force is towards the region with fewer field lines (i.e less flux density).




On one side of the conductor, the magnetic fields oppose each other and some cancel out resulting in formation of a relatively weaker field there.

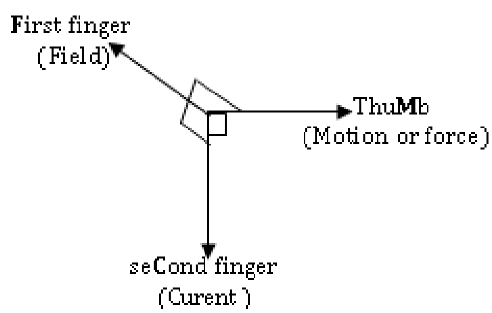
On the other side of the conductor, the applied magnetic field lines are forced to curve or concentrate resulting in formation of a strong magnetic field there.
There are more field lines above the wire since both fields act in the same direction.

A force is therefore exerted on the conductor that moves it from a region of strong magnetic field to a weaker magnetic field.

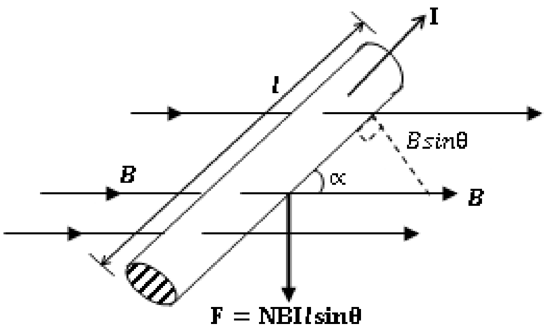
If we suppose field lines to be a stretched elastic material, these below will try to straighten out and in so doing will exert a down ward force on the wire.
[See the kicking wire experiment for verification]
[See UNEB 2003 Qn.5 (d)]

**Fleming's Left Hand Rule (Motor rule)**
The direction of the force in a current carrying conductor placed across a magnetic field is predicted by the Fleming's Left Hand Rule.

It states that if the thumb, first and second fingers of the left hand are held mutually at right angles with the thuMb pointing in the direction of magnetic force (or Motion), the First finger indicates the direction of the field while the seCond finger indicates the direction of current in the conductor.




(b) Factors affecting the magnitude of a force on a current carrying conductor.
Experiments show that the magnitude of the force exerted is proportional to the:-
(i) Current I in the conductor
(ii) Length, *l*, of the conductor
(iii) Strength of the magnetic field by a quantity called magnetic flux density, B.
(iv) Number of turn in the conductor.
(iv) $\sin\theta$, where $\theta$ is the angle between the conductor and the magnetic field.




$\mathbf{F} \propto \mathbf{NBI}\boldsymbol{l}\,\mathbf{sin} \propto$
$\mathrm{F} = \mathrm{kNBI}l \sin \propto$
Where k = constant o rotation.

When B =1T then F = 1N, I = 1A, $l$ = 1m, N=1 (For a straight wire), $\propto = 90^0$; then, from,
$\mathrm{F} = \mathrm{kNBI}l \sin \propto$
1 = k x1 x 1 x 1 x 1x1
k = 1
Therefore:
$\mathbf{F} = \mathbf{NBI}\boldsymbol{l}\,\mathbf{sin} \propto$

When the conductor is parallel to the field, $\propto = 0^0\ thus\ F = NBIl\,sin\,0^0 \Leftrightarrow \boldsymbol{F} = 0$

When the conductor is perpendicular to the field, $\propto = 90^0\ thus\ F = NBIl\,sin\,90^0 \Leftrightarrow \boldsymbol{F} = \boldsymbol{NBIl}$

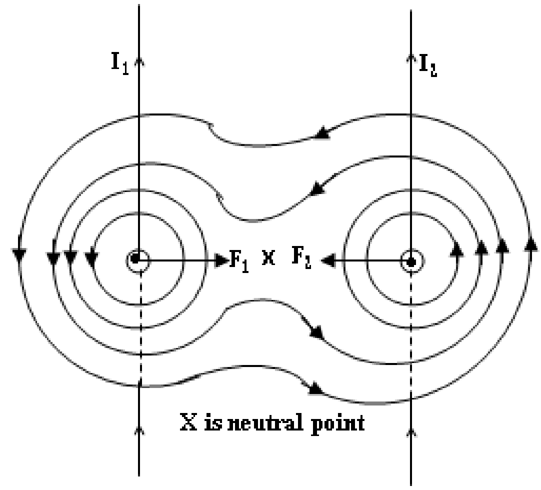
Magnetic flux density, B is the force that acts on a straight conductor (N=1turn) of length 1m, carrying a current of 1A at right angles to the magnetic field.

$$\mathrm{B} = \frac{\mathrm{F}}{\mathrm{NBIl}}$$

The unit of B is the Tesla (T)
The Tesla is the magnetic flux density in a magnetic field when the conductor of length 1m, carrying a current of 1A, placed at right angles to the magnetic field experiences a force of 1N.

**Force between two straight conductors carrying current in a vacuum**
Two current carrying conductors (wires) exert a force on each other due to the interaction of the magnetic fields set up around each conductor.
Depending on the direction of the currents, $\mathrm{I}_1$and $\mathrm{I}_2$ in the two conductors, the force exerted can be;
(i) Attractive (Same direction of $\mathrm{I}_1$and $\mathrm{I}_2$)
(ii) Repulsive (Different directions of $\mathrm{I}_1$and $\mathrm{I}_2$)

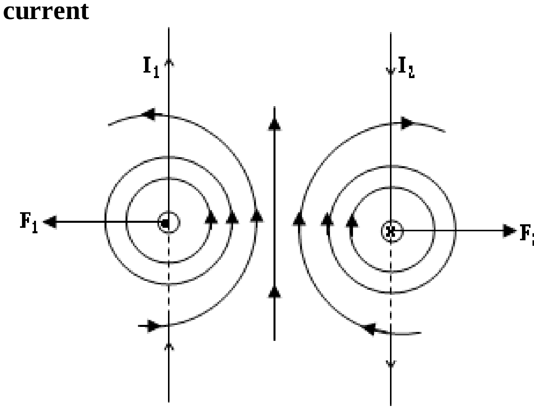
**Magnetic field due to two straight wires carrying current in the same direction**




The fields in the middle of the conductors are in opposite directions. Hence they attract each other.

A force on each wire acts from a region of strong field hence straight parallel wires carrying current in the same direction attract. i.e. "like currents attract"

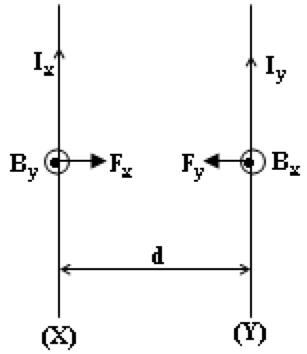
**Magnetic field due to straight wires carrying unlike current**




"Unlike currents repel"
The fields in the middle of the conductors are in the same direction. Hence they repel each other.

A force on each wire acts from a region of strong field hence straight parallel wires carrying current in opposite direction repel. i.e. "Unlike currents repel"

**Magnitude of the force between two wires carrying current.**
Consider two long parallel conductors X and Y of negligible cross sectional area, lengths,$l_x\ and\ l_y$, carrying currents $I_x\ and\ I_y$ respectively and separated by a distance, d.as shown bellow.




Each conductor has a force on it due to the magnetic field of the other.

- Wire, Y experiences a force $F_y$ due to the magnetic field of wire, X.

- Thus; $F_y = B_X I_y l_y$ $\quad But \quad$ $B_X = \frac{ì_0 I_x}{2ðd}$

$$F_y = \left(\frac{ì_0 I_x}{2ðd}\right) I_y l_y$$

$$F_y = \frac{ì_0 I_x I_y l_y}{2ðd}$$

- Similarly, Wire, X experiences a force $F_x$ due to the magnetic field of wire, Y.

- Thus; $F_X = B_y I_x l_x$ $\quad But \quad$ $B_y = \frac{ì_0 I_y}{2ðd}$

$$F_x = \left(\frac{ì_0 I_y}{2ðd}\right) I_x l_x$$

$$F_x = \frac{ì_0 I_y I_x l_x}{2ðd}$$

If the two conductors are of unit length, i.e, $l_x = l_y = 1\text{m}$, then the force per metre (or force per unit length) of the conductor is given by;

$$\boldsymbol{F} = \frac{ì_0 \boldsymbol{I_y I_x}}{2ð\boldsymbol{d}}$$

**Definition of the ampere.**

The result of the force between two parallel unit length current carrying conductors is used to define an ampere. From; $F_y = \frac{ì_0 I_y I_x}{2ðd}$ .

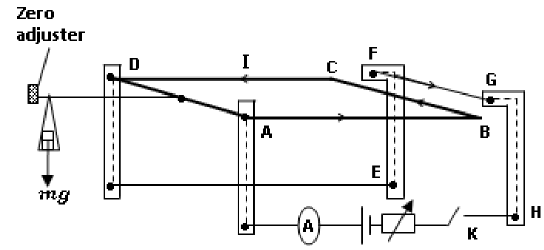
Substituting,

$ì_0 = 4ð \times 10^{-7}$ , $I_x = I_y = 1\text{A}, d = 1\text{m}$ , then;

$$F = \frac{(4ð \times 10^{-7})(1)(1)}{2ð(1)} = 2 \times 10^{-7}\text{Nm}^{-1}$$

*Hence the **ampere** is the steady current which produces a force of 2x10^{-7} Nm^{-1} between two straight and parallel conductors of infinite length and negligible cross section area separated by 1m in a vacuum.*

**Absolute measurement of current using the current balance**




The apparatus is arranged so that the distance of the scale pan from AD is equal to the distance of BC from AD.

If***l***is the length of FG and BC and ***d*** is the separation of FG and BC when I is zero (0A). The separation *d* is measured using a traveling microscope.

With no current flowing the zero adjuster is adjusted until plane ABCD is horizontal.

The switch, K is closed, so that current I flows through ABCD and EFGH in series.

The conductors FG and BC experience a repulsive force, F and BC moves down wards.

Masses m are added to the scale pan to restore balance until ABCD is horizontal again. The total weight mg added is noted.

At equilibrium,

$$\text{Magnetic Force} = \text{Weights to restore balance}$$

$$\frac{ì_0 I^2 l}{2ðd} = mg$$

$$I^2 = \frac{2ðdmg}{ì_0 l}$$

$$\boldsymbol{I} = \sqrt{\left(\frac{2ð\boldsymbol{dmg}}{ì_0 \boldsymbol{l}}\right)} = \sqrt{\left(\frac{dmg}{2 \times 10^{-7} l}\right)}$$

***Advantage over ordinary pointer ammeter***

- it gives an accurate method because it involves measurement of fundamental quantities of length and mass.

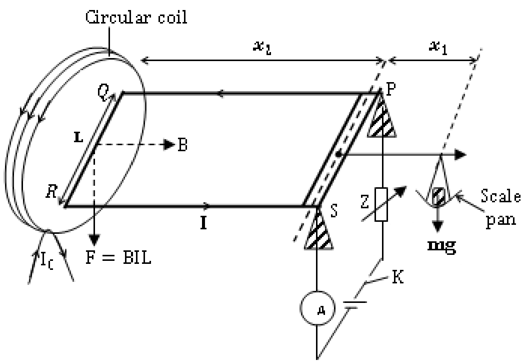
***Disadvantages***

- It is not portable

- It doesn't give direct readings and requires a skilled person.

**Experiment to investigate the dependence of magnetic flux density, B at the centre of a circular coil on the current, I through the coil using an ampere balance.**

A rigid rectangular copper frame PQRS is arranged so that one of its shorter sides QR is at the centre of a circular coil of N-turns and radius r.




When switch K is open and current, $I_C$ passed into the coil, a magnetic field B is set up at the centre of the coil in accordance with the right hand grip rule.

The rectangular copper frame PQRS is made horizontal by sliding the scale pan along a wire attached to PS.

Switch K is then closed and a current I made to flow through the frame PQRS in such a way that a magnetic force, $F = BIL$ acts vertically downwards.

The current, I is then adjusted using a rheostat Z until a reasonable downward deflection on QR is obtained.

Small weights of known masses are added onto the scale pan until the frame PQRS balances horizontally again.

The total weight in the scale pan, **mg** is noted.

Distances $PQ = \mathbf{L}$, $\boldsymbol{x_1}$, and $\boldsymbol{x_2}$ are measured using a metre- rule and noted.

The ammeter reading, **I** is read and recorded.

At equilibrium;

$$\text{Moment of F about SP} = \text{Moment of mg about SP}$$

$$F \times x_1 = mg \times x_2$$

$$BILx_1 = mgx_2$$

$$\boldsymbol{B = \frac{mgx_2}{ILx_1}}$$

**Note:**

*The above experiment can be described to investigate the relationship between the force on a current carrying conductor in a uniform magnetic field and the current.*

*The experiment is repeated for various values of current, I ant the results tabulated.*

*A graph of mg against I is plotted. It is a straight line graph through the origin. Thus,* $\boldsymbol{mg \propto I}$

**Example: 1**

Two parallel wires each of length 75cm are placed 1.0cm apart. When the same current is passed through the wires, a force of 5.0x $10^{-5}$N develops between the wires. Find the magnitude of the current.

**Solution:**

$$F = \frac{\text{ì}_0 I_1 I_2 l}{2\text{ð}d}$$

$$5 \times 10^{-5} = \frac{(4\pi \times 10^{-7})(I^2)(75 \times 10^{-2})}{2\pi(1 \times 10^{-2})}$$

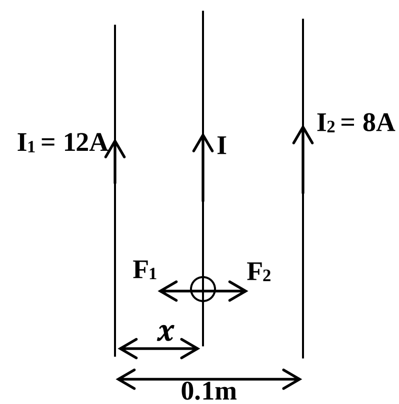
$$I = 3.33A$$

**Example: 2**

(i) Sketch the magnetic field due to two long parallel conductors carrying respective currents of 12A and 8A in the same direction. (***See notes***)

(ii) If the wires are 10cm apart, find where a third parallel wire also carrying a current must be placed so that the force it experiences is zero.

**Solution:**




Let the third wire carry current **I** and be **x m** from wire carrying a current of 12A.

$F_1$ is force exerted on the wire due to current 12A, $F_2$ is the force exerted on wire due to current of 8A.

$$F = \frac{\text{ì}_0 I_1 I_2 l}{2\text{ð}d}$$

$$F_1 = \frac{\text{ì}_0(12)Il}{2\text{ð}x}, F_2 = \frac{\text{ì}_0(8)Il}{2\text{ð}(0.1 - x)}$$

If net force is zero, then;

$$F_1 = F_2$$

$$\frac{\text{ì}_0(12)Il}{2\text{ð}x} = \frac{\text{ì}_0(8)Il}{2\text{ð}(0.1 - x)}$$

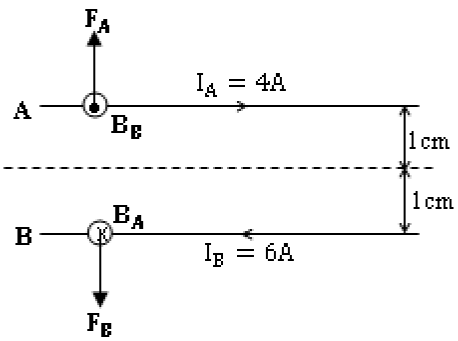
$$\frac{12}{x} = \frac{8}{(0.1 - x)}$$

$$x = 0.06 \text{ m}$$

**Example: 3**

Two long straight wires of A and B carry currents of 4A and 6A respectively in a vacuum. If A and B are parallel to each other and are a distance of 2.0cm apart, calculate the resultant magnetic field mid way between the wires carrying currents in;
(i) opposite directions.
(ii) the same direction.

**Solution:**
**(i)** Opposite directions.




$B_A = \frac{\text{ì}_0 I_A}{2\text{ð} d_A}$ ; $B_B = \frac{\text{ì}_0 I_B}{2\text{ð} d_B}$

The resultant magnetic field, B is given by;

$\mathrm{B} = \mathrm{B_A} + \mathrm{B_B}$

$\mathrm{B} = \frac{\text{ì}_0 I_A}{2\text{ð} d_A} + \frac{\text{ì}_0 I_B}{2\text{ð} d_B}$

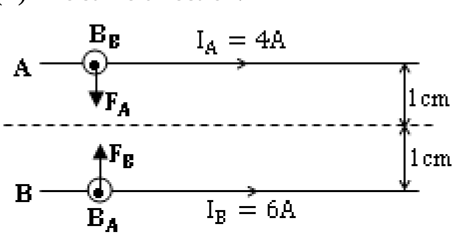
Where, $d_A = d_B = d$

$\mathrm{B} = \frac{\text{ì}_0}{2\text{ð} d}(I_A + I_B)$

$\mathrm{B} = \frac{(4\text{ð} \times 10^{-7})}{2\text{ð}(1 \times 10^{-2})}(4 + 6)$

$\mathrm{B} = 2 \times 10^{-4}\mathrm{T}$ Perpendicularly into the plane of the paper.

(ii) The same direction.




$B_A = \frac{\text{ì}_0 I_A}{2\text{ð} d_A}$ ; $B_B = \frac{\text{ì}_0 I_B}{2\text{ð} d_B}$

The resultant magnetic field, B is given by;

$\mathrm{B} = \mathrm{B_A} + (-\mathrm{B_B})$

$\mathrm{B} = \frac{\text{ì}_0 I_A}{2\text{ð} d_A} - \left(\frac{\text{ì}_0 I_B}{2\text{ð} d_B}\right)$

Where, $d_A = d_B = d$

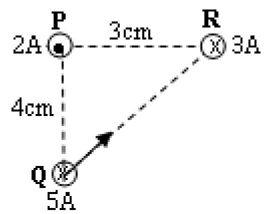
$\mathrm{B} = \frac{\text{ì}_0}{2\text{ð} d}(I_A - I_B)$

$\mathrm{B} = \frac{(4\text{ð} \times 10^{-7})}{2\text{ð}(1 \times 10^{-2})}(4 - 6)$

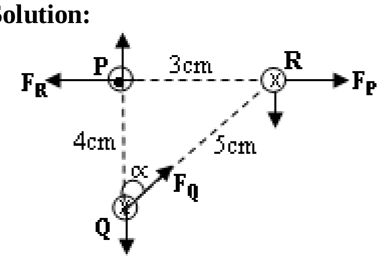
$\mathrm{B} = -4 \times 10^{-5}\mathrm{T}$

$|\mathrm{B}| = 4 \times 10^{-5}\mathrm{T}$ Perpendicularly out of the plane of the paper.

**Example: 4**
Three parallel and infinitely long wires P, Q and R are arranged at the corners of a triangle of sides 3cm by 4cm by 5cm as shown in the figure below.




Find the force per metre on the wire R.
**Solution:**




$F_P = \frac{\text{ì}_0 I_P I_R l}{2\text{ð} d}$

The force per metre on R due to P is thus:

$F_P = \frac{F_P}{l} = \frac{\text{ì}_0 I_P I_R}{2\text{ð} d} = \frac{(4\text{ð} \times 10^{-7})(2)(3)}{2\text{ð}(3 \times 10^{-2})}$

$F_P \qquad = 4 \times 10^{-5}\mathrm{Nm^{-1}}$

Away from R along PR.

The force per metre on R due to Q is thus:

$F_Q = \frac{F_Q}{l} = \frac{\text{ì}_0 I_Q I_R}{2\text{ð} d} = \frac{(4\text{ð} \times 10^{-7})(5)(3)}{2\text{ð}(3 \times 10^{-2})}$

$F_P \qquad = 6 \times 10^{-5}\mathrm{Nm^{-1}}$

Away from R along QR.

Resolving the forces: Horizontally;

$\sum F_x = F_P - F_Q \cos \text{è}$

$= (4 \times 10^{-5}) - (6 \times 10^{-5}) \cos 53.1^0$

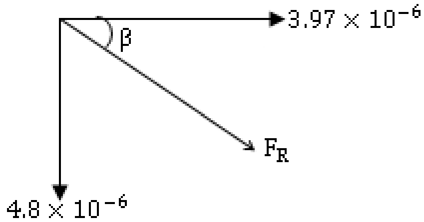
$= 3.97 \times 10^{-6}\mathrm{Nm^{-1}}$

Resolving the forces: Vertically;

$\sum F_y = -F_Q \sin$ è

$= -(6 \times 10^{-5}) \sin 53.1^0$

$= -4.8 \times 10^{-6}\text{Nm}^{-1}$




$F_R^2 = F_x^2 + F_y^2$

$F_R^2 = (3.97 \times 10^{-6})^2 + (-4.8 \times 10^{-6})^2$

$F_R = 4.82 \times 10^{-5}\text{Nm}^{-1}$

$\tan$ â $= \frac{F_y}{F_x}$

â $= \tan^{-1}\left(\frac{F_y}{F_x}\right)$

â $= \tan^{-1}\left(\frac{-4.8 \times 10^{-6}}{3.97 \times 10^{-6}}\right)$

â $= 85.3^0$

**Example: 5**

A current of 1.0A flows in a long solenoid of 100 turns per meter. If the solenoid has a mean diameter of 80cm, find the magnetic flux linkage on one meter length of the solenoid.

**Solution:**

$B = \frac{ì_0 NI}{l} = ì_0 nI = (4ð \times 10^{-7})(100)(1)$

$B = \quad 1.275 \times 10^{-3}$ T

Magnetic flux linkage, $\phi$

$\phi = AB \cos$ è

$\phi = ð(80 \times 10^{-3})^2(1.275 \times 10^{-3})$

$\phi = 6.317 \times 10^{-1}\text{Wb}$

4. A small circular coil of 10 turns and mean radius of 2.5cm is mounted at the center of a long solenoid of 750turns per metre. If the current in the solenoid is 2.0A, Calculate:-

(i) The magnetic flux density inside the solenoid.

(ii) The initial torque on the circular coil when a current of 1.0A is passed through it.

(i)

$B = \frac{ì_0 NI}{l} = ì_0 nI = (4ð \times 10^{-7})(750)(2.5)$

$B = \quad 1.88 \times 10^{-3}$ T

(ii)

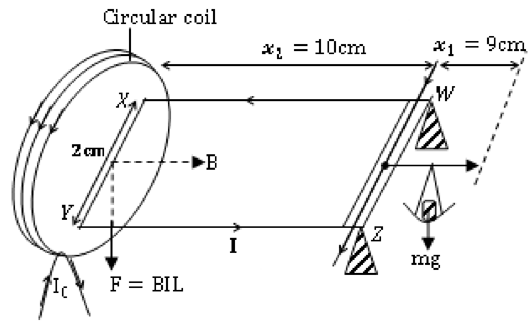
$T = BINA \cos$ è

$T = (1.88 \times 10^{-3})(1)(10)(2.5 \times 10^{-2})^2$

$T = 3.7 \times 10^{-5}\text{Nm}$

**Example: 6 UNEB 1998 No. 5 (c) (ii):**

A rectangular loop of wire WXYZ is balanced horizontally so that the length XY is at the centre of a circular coil of 500 turns of mean radius 10.0cm as shown in the figure below.




.When a current I is passed through XY and the circuit coil, a rider of mass $5.0 \times 10^{-4}$ kg has to be placed at a distance of 9.0cm from WZ to restore balance. Find the value of the current I.

**Solution:**

At horizontal equilibrium;

$\text{Moment of F about WZ} = \text{Moment of mg about WZ}$

$F \times x_1 = mg \times x_2$

$BILx_1 = mgx_2$

$\left(\frac{ì_0 NI}{2R}\right) ILx_1 = mgx_2$

$\left(\frac{ì_0 N}{2R}\right) I^2 Lx_1 = mgx_2$

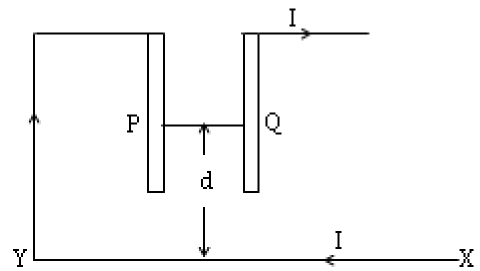
$\left[\frac{4ð \times 10^{-7} \times 500}{2 \times (10 \times 10^{-2})}\right](2 \times 10^{-2})(10 \times 10^{-2})I^2$
$= (5.0 \times 10^{-4} \times 9.81) \times (9.0 \times 10^{-2})$

$9.81 \times 4.5 \times 10^{-5} = (6.283 \times 10^{-6})I^2$

$I = 8.38$ A

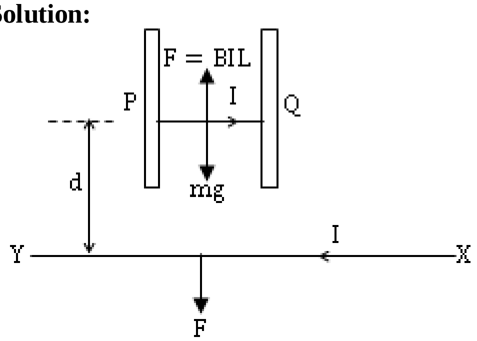
**Example: 7**

A wire XY rests on a horizontal non conducting table and another wire PQ of length 12.0cm is free to move vertically in the guides provided at the ends P and Q above the wire XY as shown in the diagram.




The mass per unit length of wire PQ is $3\text{mgcm}^{-1}$. A current of 3.6A passing through the wire is sufficient enough to maintain wire PQ a distance $\mathbf{d}$ cm above XY. Calculate the;
(i) distance of separation of the wires, d.
(ii) Magnetic flux density at the point from wire PQ due to wire XY.
(iii) Magnetic force experienced by wire PQ (Take $g = 9.8\text{Nkg}^{-1}$).

**Solution:**




$m \ = 3\text{mgcm}^{-1} \times 12\text{cm}$
$m \ = 36\text{mg}$
$m \ = 36 \times 10^{-6}$ kg

At equilibrium:
$\text{Upward magnetic force on wire PQ due to wire XY} = \text{Weight of the wire PQ at equillibrium.}$

$$F = mg$$

$BIL \ = mg$
$\left(\frac{\mu_o I}{2\pi d}\right) IL \ = mg$
$\left(\frac{\mu_o}{2\pi d}\right) I^2 L \ = mg$

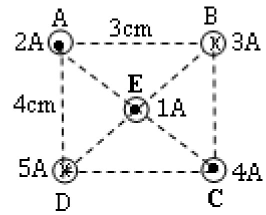
$\left(\frac{4\pi \times 10^{-7}}{2\pi d}\right)(2 \times 10^{-2})(3.6)^2 \ = (36 \times 10^{-6}) \times 9.8$
$d = 8.816 \times 10^{-4}\text{m}$

(ii)
$B = \frac{\mu_o I}{2\pi d}$

$B = \frac{(4\pi \times 10^{-7})(3.6)}{2\pi(8.816 \times 10^{-4})}$
$B = 8.167 \times 10^{-4}$ T

(iii)
$F = BIL$
$F = (8.167 \times 10^{-4}) \times (3.6) \times (12 \times 10^{-2})$
$F = 3.528 \times 10^{-4}\text{N}$

**Exercise**
1. The figure above shows a wire arranged at the corners of a rectangle ABCD of sides 4cm by 3cm. Wire E at the centres of the diagonals AC and BD carries a current of 1A.

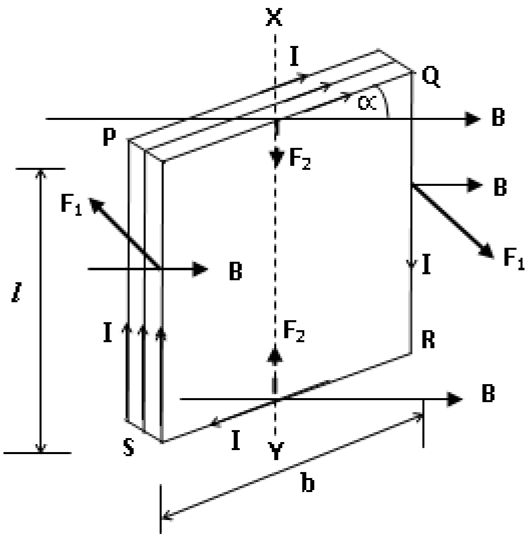



Find the resultant force per metre of wire E.
$[F_R = 1.92 \times 10^{-5}\text{Nm}^{-1}\text{towards the west}]$

2. Find the net force on a rectangular frame of 20cm by 5cm carrying a current of 5A, when side of the frame is placed 5cm from a long straight wire carrying a current of 20A.$[\text{F}_{\text{net}} = 4 \times 10^{-5}\text{N}]$

3. UNEB: 2000 Qn. 6 (d).
UNEB: 2001 Qn. 5 (a) (ii).
UNEB: 2002 Qn. 7 (a) (ii).
UNEB: 2002 Qn. 6 (a) (iii).$[\text{F}_\text{A} = 8 \times 10^{-5}\text{N}]$
UNEB 2003 Qn. 7 (a) (ii)$\left[\text{F}_\text{Q} = 4 \times 10^{-7}\text{N to P}\right]$
UNEB: 2006 Qn. 6 (c)
UNEB: 2007 Qn. 5 (b)
UNEB: 2009 Qn. 5 (c) (ii).$[x = 4.8\text{cm}]$

**Torque on a current carrying coil in a uniform magnetic field**
Consider a rectangular coil PQRS of N turns carrying a current I, in a uniform magnetic field of flux density B. Suppose initially the plane of the coil makes an angle ∝ with the magnetic field as shown bellow.

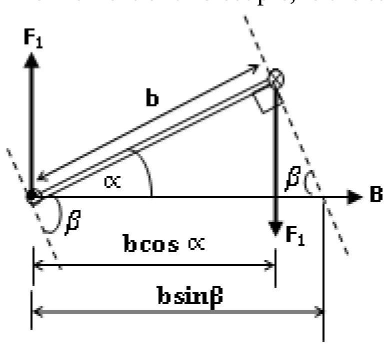



By Fleming's left hand rule;
The force on the side PQ of the coil is;
$F_2 = \mathrm{NBI}l \sin \propto$, vertically downwards, (i)
The force on side RS of the coil is.
$F_2 = \mathrm{NBI}l \sin \propto$, vertically up wards,………. (ii)

The forces on PQ and RS,$F_2$compress the coil and are resisted by the coil's rigidity. Hence they cancel out.

The force on the side SP of the coil is;
$F_1 = \mathrm{NBI}l$, perpendicularly into the plane of the paper,……………………………………… (iii)
The force on side QR of the coil is
$F_1 = \mathrm{NBI}l$, Perpendicularly out of the plane of the paper……………………………………… (iv)

The forces on SP and QR are equal and opposite and hence constitute a couple which rotates the coil about the axis XY. The moment of this couple, is the torque.




$\text{Torque of the coil} = \text{One of the forces} \times \text{Perpendicular distance between them.}$

**Torque** of the coil, $\hat{o} = \mathbf{F_1} \times \boldsymbol{bcos} \propto$

Torque of the coil, $\hat{o} = \mathrm{NBI}l \times bcos \propto$

$\hat{o} = \mathrm{NBI}(l \times b)cos \propto$

But, $l \times \mathrm{b} = \mathrm{A}$, the cross − sectional area of coil

$\hat{o} = \mathrm{NBI}(A)cos \propto$

$$\tau = \mathrm{NABI}cos \propto$$

If â is the angle between the field and the normal to the plane of the coil, then;

$è + â = 90^0$

$è = 90^0 - â$

Thus;

$\tau = \mathrm{NABI}cos(90^0 - â)$

$\tau = \mathrm{NABI}sinâ$

The units of torque are newton metre, (Nm).

The coil rotates through an angle θ until it's stopped by the restoring torque (opposing torque) of the torsion wire.

$$\tau_R \propto \theta$$
$$\tau_R = \mathrm{k}\theta$$

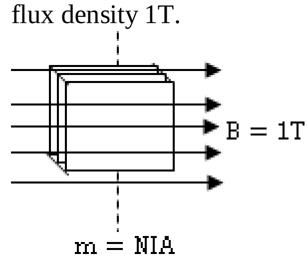
At equilibrium;

Deflecting Torque, $\tau_m$ = Restoring torque, $\tau_R$

$\mathbf{NABIcos} \propto\ = \mathbf{k}è$

**Magnetic moment of the coil. m**

The magnetic moment of a coil is the torque exerted on a coil when it is placed with its plane parallel to a field of magnetic flux density 1T.




Plane of the coil is parallel, $\Leftrightarrow \propto = 0^0$

Magnetic flux density of the coil, B = 1T

Magnetic moment of the coil, $\mathrm{m} = \mathrm{NABI}cos \propto$

$$\mathrm{m} = \mathrm{NA}(1)\mathrm{I}cos(0^0)$$

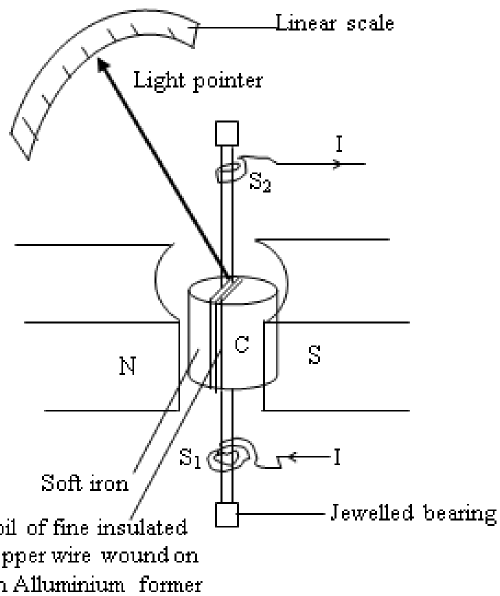
$\mathbf{m = NAI}$

The quantity, $\mathrm{m} = \mathrm{NAI}$ is called the magnetic moment of the coil.

**MOVING COIL GALVANOMETER**

Structure:

A moving coil galvanometer consists of a rectangular coil of fine insulated copper wire wound on an Alluminium frame and pivoted by means of jewel bearing in a radial magnetic field provided by concave soft iron pieces of a powerful permanent magnet. In between the pole pieces lies a cylindrical core C of soft iron.




$S_1$ and $S_2$ are hairs springs that provide a resisting torque on the coil and from inlets and outlets for current to be measured.

Mode of operation:
It works on the principle of force on a current carrying conductor.
The current I, to be measured is passed through the coil via spring S1 to S2. The coil experiences a couple whose torque is given by;
$\tau_m = \text{NABI}$; Since $\propto = 0^0$ for radial fields.

This torque rotates the coil until the restoring torque, due to twist in the springs is equal to that due to the magnetic field.
The restoring torque is directly proportional to the deflection $\theta$ of the coil. Hence restoring torque is given by;
$\tau_R = k\theta$:

At equilibrium;
$$\text{Deflecting Torque}, \tau_m = \text{Restoring torque}, \tau_R$$
$$\text{NABI} = k\theta$$
$$\text{I} = \left(\frac{k}{\text{NAB}}\right)\theta$$

$\Leftrightarrow$ $\text{I} \propto \theta$: Hence a linear scale.

$\theta$ = Anle turned trough by the pointer
k = spring constant or torsion constant whose units are (**Nmdiv$^{-1}$**)

Current sensitivity
This the deflection per unit current, $\frac{\theta}{\text{I}}$
From; $\text{NABI} = k\theta$

$\frac{\theta}{\text{I}} = \frac{\text{NAB}}{k}$

A galvanometer is said to be sensitive if it shows a large deflection when a small current passes through it. There fore, to increase the sensitivity of the galvanometer;
Increase the number of turns.
Use strong permanent magnets.
Use a coil of large area.
Use weaker or very fine hair springs which can easily turn (i.e. k must be very low or low rigidity modulus).
Use thick copper wires of low resistance for making the coil.

Voltage sensitivity
This the deflection per unit voltage, $\frac{\theta}{\text{V}}$
From; $\text{NABI} = k\theta$
; $\text{NAB}\left(\frac{\text{V}}{R}\right) = k\theta$

$\frac{\theta}{\text{V}} = \frac{\text{NAB}}{kR}$

**Note**
1: Since large A and N imply a bigger and heavier coil, and yet a large B implies a small gap between the poles of the permanent magnet, a compromise must be sought.

2: The magnetic field is made radial so as to obtain a linear scale. This is achieved by making the poles semi-circular (or concave) and placing the soft iron cylinder at the center of the field or between the poles.
The soft iron also helps to concentrate the field.
The coil in a radial field experiences a maximum torque, $\text{T} = \text{BINA}$.
A radial field is that field in which the plane of the coil in all positions remains parallel to the direction of the magnetic field.
$\Leftrightarrow$ $\text{I} \propto \theta$: Hence a linear scale.

3: The coil is wound on a metal frame so that eddy currents induced in the coil when it is in motion cause damping, thus making the instrument dead beat.
Dead beat means that coil comes to rest in the shortest time possible.
If a coil just moves to its final position without oscillating about it, it is said to have a dead beat action.
The Alluminium frames help damp oscillations of the coil so that the coil doesn't oscillate.
A moving coil galvanometer is designed to be critically damped.

Eddy currents are formed when a conductor is rotated in a uniform field or placed in a changing field. These currents are useful because they cause electromagnetic damping which enables the pointer to settle quickly.
Electro magnetic damping is the opposition to the motion of a conductor in a magnetic field due to eddy currents induced in the conductor by a changing magnetic flux linking it.

Conversion of a moving coil galvanometer into a ballistic galvanometer.
A ballistic galvanometer responds to a current pulse or an electric blow. Thus when a current is passed into the instrument, all of it or all the charge must pass into the coil at once before the coil moves appreciably.
Hence it responds to a large quantity of inertia.

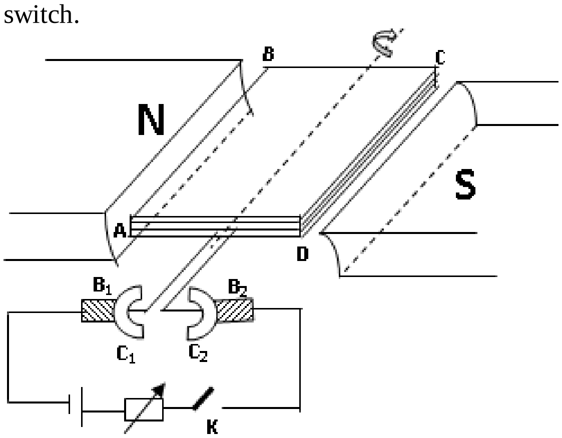
Necessary modifications:
In order to convert a moving coil galvanometer into a ballistic galvanometer;
(i) Minimise electromagnetic damping by replacing the Alluminium frame with an insulting frame.
(ii) Increase the inertia of the coil by increasing the number of turns, N of the coil hence making the coil heavier.
(iii) Increase the period, T of oscillation of the coil by using a fine and very flexible wire as the suspension wire of low torsion constant, k of low rigidity modulus.
(iv) Minimise the restoring torque by using extremely fine wire hair springs.
Applications of a ballistic galvanometer.
A ballistic galvanometer is used to;
Measure charge induced momentarily, hence the flux and magnetic density

$$Q \propto \theta \Leftrightarrow Q = \frac{N\Phi}{R} \Leftrightarrow Q = \frac{N(BA)}{R}$$

Where $\theta$ is the deflection of the B.G.
Investigate the magnetic field of a solenoid.
Compare capacitances.

**SIMPLE D.C. MOTOR**
A d.c motor is a device which converts electrical energy into mechanical energy.

Structure:
It consists of a rectangular coil of wire pivoted between the poles of a strong magnet.
The ends of the coil are connected to two halves of split rings (Commutators) which press lightly against the carbon brushes.
The coil is then connected in series with a battery and a switch.




How the motor works.
When K is closed current is passed thru the coil in the direction shown.

Applying Fleming's left hand rule, AB experiences an upward force while CD experiences a down ward force. The two forces exerted on the coil are equal and opposite hence they constitute a couple which rotates the coil in a clockwise direction.

When the coil reaches the vertical position, the brushes $B_1$ and $B_2$ lose contact with the Commutators $C_1$ and $C_2$ hence the current is cut off.

The momentum of the coil however carries it beyond the vertical position and the Commutators halves reverse contact and the coil continues to rotate in the same direction.

**Note:**
When a conductor (coil) is moved in a magnetic or placed in a varying magnetic field, the magnetic flux linked with it changes and hence an e.m.f in opposite direction will be induced it.
This e.m.f is called back e.m.f. The currents induced are called eddy currents.

**Back e.m.f** is the e.m.f induced in a conductor when it rotates in a magnetic field which tends to oppose the motion of the conductor.

**Eddy currents** are currents induced in a conductor due to a changing magnetic flux linking it.

If V is the supply voltage and E is the back e.m.f, then the current flowing in the coil, $\boldsymbol{I_a}$ is given by;

From the principle of conservation of energy;

$$\text{Power supplie} = \text{Mechanical power developed} + \text{Power lost in the windings.}$$

$$\boldsymbol{VI = EI + I^2 R_a}$$

$$\boldsymbol{I_a = \frac{V-E}{R_a}};$$

Where $R_a$ is the armature resistance.

**Importance of Back e.m.f.**
- It increases the efficiency by increasing the mechanical power or speed of rotation since back e.m.f is proportional to the speed of rotation of the motor.

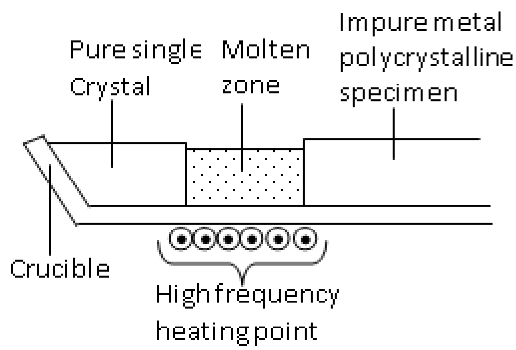
- It minimizes power losses or heat dissipated in the windings which would otherwise burn the coil limiting the current flowing in the coil (armature windings).

NOTE:
From the motor equation, $\boldsymbol{I_a = \frac{V-E}{R_a}}$ ; the bigger the back e.m.f, the smaller the current and hence $I^2R$ loses are minimized hence increasing the efficiency.

**Uses of Eddy currents**

- They cause electromagnetic damping which helps the coil to come to rest quickly in moving coil galvanometers, car speed meters, e.t.c.
- They are used in electromagnetic brakes.
- They are used in detection of cracks in metals
- They are used in sorting materials from solid wastes (used in induction furnace during refining of metals).

<u>The Induction furnace</u>




It is used in refining of metals. A heating coil is fed with alternating current.
The rapidly changing magnetic flux links up the specimen material and eddy currents are induced.
The internal energy produced cause the material to melt. The impurities collect in the molten zone.

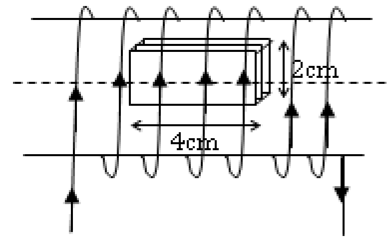
**Disadvantages of Eddy currents**

- For transformers and motors: They cause un necessary heating hence power losses.
- For ballistic galvanometers: They cause electromagnetic damping which is un desirable in a ballistic galvanometer.

**Reasons why the efficiency of a motor is always less than 100%.**

- Energy losses due to friction
- Internal energy loss due to resistance of the windings.
- Energy loss due to lifting useless loads
- Energy loss due to eddy currents

**Examples: 1**
A small rectangular coil of 10 turns and dimensions 4cm by 20cm is suspended inside a long solenoid of 1000 turns per metre so that its frame lies along the axis of the solenoid as shown in the figure below.




The coil is connected in series with the solenoid. The coil deflects through $30^0$ when a current of 2.0A is passed through the solenoid. Find the torsion constant of the suspension wire.

**Solution:**

$\mathrm{I_{coil}} = \mathrm{I_{solenoid}} = 2.0\mathrm{A}$
$\mathrm{N_{coil}} = 10\mathrm{turns}$
$$\mathrm{n_s} = \frac{\mathrm{N_s}}{\mathrm{L}} = 1000\text{turns per metre.}$$

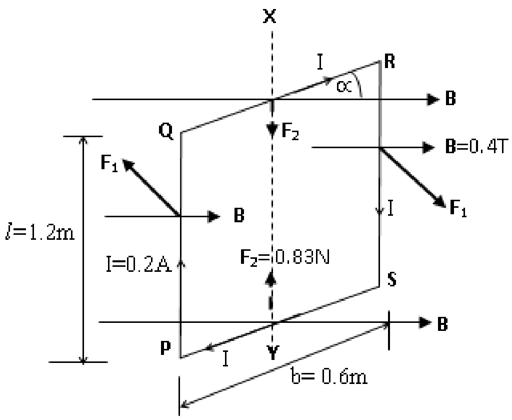
$\mathrm{A_{coil}} = l \times w$
$$\mathrm{A_{coil}} = \frac{4}{100} \times \frac{2}{100}$$
$A_{coil} = 8 \times 10^{-4}\mathrm{m}^2$

$B_s = \mu_0 n_s I_s$
$B_s = (4\pi \times 10^{-7}) \times 1000 \times 2$
$B_s = 2.51 \times 10^{-3}$ T

When the coil stops moving,
$\text{Deflection torque} = \text{Restoring torque}$
$\mathrm{BANI}\cos\alpha = \mathrm{k}\theta_0$

Initially, the plane of the coil is parallel to B. Hence $\alpha = 0$
$$\theta_0 = 30^0 = \frac{30\pi}{180} = \frac{\pi}{6}\ rad$$
$\mathrm{B}_c\mathrm{A}_c\mathrm{N}_c\mathrm{I}_c\cos\alpha = \mathrm{k}\theta_0$
$$(2.51 \times 10^{-3})(8 \times 10^{-4}) \times 10 \times 2\cos 0 = \mathrm{k}\left(\frac{\pi}{6}\right)$$
$\mathrm{k} = 7.67 \times 10^{-5}\mathrm{Nmrad}^{-1}$

**Examples: 2**
The figure below shows a rectangular coil PQRS of sides $1.2m \times 0.6m$ carrying a current of 0.2 A in a magnetic field of flux density 0.4T.




If side PS experiences a force of 0.83N, find the;
(i) Angle á
(ii) Torque exerted on the coil
(iii) Magnetic moment of the coil.

**Solution:**
(i) $\text{Number of turns}, \mathrm{N} = 80 \text{ turns.}$
$\mathrm{F_{PS}} = \mathrm{NBI}b\sin\propto$
$0.83 = 80 \times 0.4 \times 0.2 \times 0.6 \sin\propto$

$$\sin\propto = 0.2161$$

$\propto = 12.5^0$

(ii) The torque exerted on the coil, T
$\mathrm{A_{coil}} = l \times w$
$\mathrm{A_{coil}} = 1.2 \times 0.6$
$\mathrm{A_{coil}} = 0.72 \text{ m}^2$

$\mathrm{T} = \mathrm{BANI}\cos\alpha$
$\mathrm{T} = 80 \times 0.72 \times 0.4 \times 0.2 \cos 12.5^0$
$\mathrm{T} = 4.499 \text{ Nm}$

(iii) Magnetic moment of the coil, m;
$m = \mathrm{NIA}$
$m = 80 \times 0.2 \times 0.72$
$m = 11.52\mathrm{Am}^2$

**Examples: 3**
(a) (i) Explain what is meant by back e.m.f of a motor.
(ii) Give the major difference between a d.c generator and a d.c. motor.
(b) A D.C motor of armature resistance 5Ω is connected to a d.c supply of 240V. If the current flowing in the coil is 4A, find the:
(i) e.m.f generated by the coil
(ii) Mechanical power developed
(iii) Power supplied to the motor
(iv) Efficiency of the motor

**Solution:**
(a) (i) and (ii) See notes.
(b)
(i) Armature resistance, $R_a = 5\Omega, I_a = 4\mathrm{A}, \mathrm{V} = 240\mathrm{V}$
From the motor equation,

$$\boldsymbol{I_a = \frac{V - E}{R_a}}$$

$4 = \frac{240 - \mathrm{E}}{5} \Leftrightarrow 240 - \mathrm{E} = 4(5)$
$\mathrm{E} = 220\mathrm{V}$

(ii) Mechanical power developed; $\mathrm{P_{mech}}$
$\mathrm{P_{mech}} = \mathrm{IE}$
$\mathrm{P_{mech}} = 4 \times 220$
$\mathrm{P_{mech}} = 880\mathrm{W}$

(iii) Power supplied to the motor
$\mathrm{P_{input}} = \mathrm{IV}$
$\mathrm{P_{input}} = 4 \times 240$
$\mathrm{P_{input}} = 960\mathrm{W}$

(iv) Efficiency of the motor
$\text{Efficiency} = \frac{\text{Mech. power developed}}{\text{Power suplied}} \times 100\%$
$\text{Efficiency} = \frac{\mathrm{P_{output}}}{\mathrm{P_{input}}} \times 100\%$

$\text{Efficiency} = \frac{\mathrm{IE}}{\mathrm{IV}} \times 100\%$

$\textbf{Efficiency} = \frac{\mathbf{E}}{\mathbf{V}} \times 100\%$

$\text{Efficiency} = \frac{220}{240} \times 100\%$
$\text{Efficiency} = 91.67\%$

**Examples: 4**
The coil of a d.c motor of resistance $5\Omega$ rotates at $80\ rev\ min^{-1}$ when connected to to a d.c supply of 240 V and a current of 10A is flowing. Find the new speed of the motor when it draws a current of 20A.

**Solution:**
Armature resistance, $R_a = 5\Omega, I_a = 10\mathrm{A}$,
$\mathrm{V} = 240\mathrm{V}, \mathrm{E} = ?$
From the motor equation,

$$\boldsymbol{I_a = \frac{V - E}{R_a}}$$

$10 = \frac{240 - \mathrm{E}}{5} \Leftrightarrow 240 - \mathrm{E} = 10(5)$
$\mathrm{E} = 190\mathrm{V}$

$\text{Speed or frequency} = 80 \text{ rev min}^{-1}$

$= \frac{80}{60}$ rev s$^{-1}$

$= \frac{4}{3}$ rev s$^{-1}$

But Back e.m.f induced is proportional to the speed of rotation of the conductor.

$E \propto f$
$E = kf$
$190 = k\left(\frac{4}{3}\right)$
$k = \frac{3}{4} \times 190 = 142.5\text{V rev}^{-1}\text{s}$

When , Armature resistance, $R_a = 5\Omega, I_a = 20\text{A}$, $\text{V} = 240\text{V}, \text{E} = ?$
From the motor equation,

$$\boldsymbol{I_a = \frac{V - E}{R_a}}$$

$20 = \frac{240 - \text{E}}{5} \Leftrightarrow 240 - \text{E} = 20(5)$
$\text{E} = 140\text{V}$

But Back e.m.f induced is proportional to the speed of rotation of the conductor.

$E \propto f$
$E = kf$
$140 = 142.5f$
$f = 0.9824\text{rev s}^{-1}$ or $f = 58.95\text{rev min}^{-1}$

**Alternatively:**

$E = k$ù
$E_1 = k$ù$_1$
$190 = k(80)$
$k = 2.375$
When , $I_a = 20\text{A}$
$E_2 = k$ù$_2$
$140 = 2.375$ù$_2$
ù$_2 = 58.95\text{rev min}^{-1}$ $or$ ù$_2 = 0.9824\text{rev s}^{-1}$

**Note:**
If the coil has N turns, each of area A m$^2$, and its rotated in a magnetic field of flux density B teslas with a uniform angular velocity, ù $rads^{-1}$, then the back e.m.f induced is;
**E = NABù sin** á

**Exercise:**

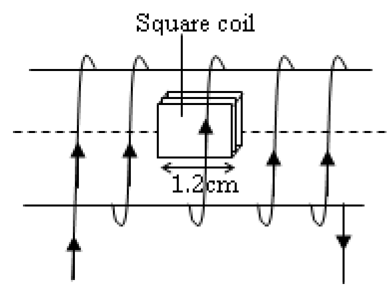
**1.** The coil of a galvanometer is 0.02m by 0.08m. It consists of 200 turns of wire and is in a magnetic field of 0.2T. The restoring torque constant of the suspension is 1x10$^{-5}$ Nm per degree. Assuming the magnetic field is radial.
(i) What is the maximum current that can be measured by the galvanometer if the scale accommodates a 45$^0$ deflection?
(ii) What is the smallest current that can be detected if the minimum observed deflection of 0.1$^0$ (7.03x10$^{-3}$A, 1.56x10$^{-5}$A)

2. A moving coil galvanometer has the following characteristics.
- Number of turns = 80
- Area o the coil = 50mm$^2$
- Flux density of the radial field = 0.2T.
- Torsional constant of the suspension wire = 5x10$^{-9}$Nm rad$^{-1}$
- Resistance of the coil = $20\Omega$

Calculate the angular deflection provided by
(i) a current of 0.01mA (1.6rad)
(ii) a p.d of 0.01mV. (0.08rad)

3. A flat circular coil of 50 turns of mean diameter 40cm is in a fixed vertical plane and has a current of 5Aflowing through it. A small coil 1cm square and having 120 turns is suspended at the center of the circular coil in a vertical plane at an angle of 30$^0$ to that o the larger coil. Calculate the torque that would act on the small coil when it carries a current of 2mA. (9.42x10$^{-9}$)

4. A square coil of side 1.2cm and with 20 turns of fine wire is mounted centrally inside and with its plane parallel to the axis of the long solenoid which has 50 turns per cm. The current in the coil is 70mA and the current in the solenoid is 6.2A.




Find the;
(i) magnetic flux density inside the coil
(ii) torque on the square coil

5. (a) A rectangular coil PQRS of 100 turns and cross sectional area$5 \times 10^{-4}\text{m}^2$ is carrying a current of 2.0A. The coil is placed in a magnetic field of flux density 0.5T such that SR is inclined at $\theta^0$ to the field. The side SR is 0.01m long and a force of 1.0N acts on it.
Calculate the;
(i) Angle$\theta$. $[\theta = 0^0]$
(ii) Torque on the coil. $[\text{T} = 5.0 \times 10^{-2}\text{Nm}]$

(b) A moving coil galvanometer of resistance $20\Omega$ gives a full scale deflection when a current of 5mA passes through it. Explain how it can be converted into an ammeter to measure a maximum current of 5A.

$$\left[\begin{array}{c} By\ connecting\ a\ shunt\ of\ resistance\ \mathrm{R_s} = 2.0 \times 10^{-2}\Omega \\ across\ it. \end{array}\right]$$

6. An electric motor taking a current of 5A at a potential difference of 240V is connected by a cable to a generator some distance away. If the potential difference at the terminals of the generator is 250V. Determine the;

(i) Resistance of the cable [$2\Omega$]

(ii) Power supplied by the generator. [$1250$W]

(iii) Power loss in the cable. [$50$W]

(iv) Efficiency of the system. [$96\%$]

7. A motor having an armature resistance of $2\Omega$ is designed to operate on 220V mains at a full speed. It develops a back e.m.f of 210V. Find the;

(i) Current in the armature when the motor is running at full speed. [$I_a = 5A$]

(ii) Current in the armature if no starter was used at beginning of the rotation of the coil. [$I_a = 110A$]

(iii) Efficiency of the motor at full speed. [$95.45\%$]

8. The coil of a d.c motor is connected to a 230V supply. If the current flowing in the coil is 5.0A and the armature resistance is 6.0Ω. Find the;

(i) E.m.f generated by the coil. [$E = 200V$]

(ii) Mechanical power developed by the motor. [$1000$W]

(iii) Power supplied to the motor. [$1150$W]

(iv) Efficiency. [$86.9\%$]

(v) Power lost. [$150$W]

## ALTERNATING CURRENT CIRCUITS

Alternating current refers to the type of current whose magnitude and direction vary continuously with time following a sine wave.

For direct current (d.c), electrons flow in only one direction but for a.c, electrons reverse direction of flow at regular intervals.

<u>Sinusoidal alternating currents and voltages</u>

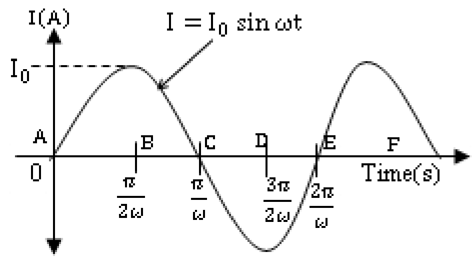
These are currents or voltages of the form

$$\mathbf{I = I_0 \sin \omega t \quad or \quad V = V_0 \sin \omega t}$$

where $I_0$ and $V_0$ are the amplitudes or peak values of current and voltage respectively.

Alternating currents or voltages vary in magnitude and direction periodically with time following a sine wave.

A graph of I against t has the form:




**Terms used in a.c circuits**

**1. Period:** Is the time taken to make a complete cycle (oscillation).

**2. Frequency, f:** Is the number of complete cycles made per second.

**3. Angular frequency, ω:** Is the rate of rotation or it is the rate of change of angle.

$$\omega = \frac{\theta}{t} \Leftrightarrow \theta = \omega t$$

For 1 complete cycle, $\theta = 2\pi$ rads, $t = T$

$\Rightarrow 2\pi = \omega T.$

$\Rightarrow 2\pi f = \omega.$

**4. Peak value (Amplitude),** $I_0$ and $V_0$

This is the maximum value of alternating current or voltage obtained over a complete cycle.

**5. Mean value (or half cycle average):**

This is the average value of current or voltage taken over a half cycle.

It is denoted by $< f(t) >_T$ and given by;

$$< f(t) >_T = \frac{1}{T}\int_0^T f(t)dt = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} f(t)dt$$

Where T is the period time, $T = \frac{2\pi}{\omega}$

$$< \cos \omega t >_T = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \cos \omega t\, dt$$

From; $\frac{d(\sin \omega t)}{dt} = \omega \cos \omega t$

$$\frac{d}{dt}\left(\frac{1}{\omega}\sin \omega t\right) = \cos \omega t$$

$$< \cos\omega t >_T = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{d}{dt}\left(\frac{1}{\omega}\sin\omega t\right) dt$$

$$< \cos\omega t >_T = \left[\frac{1}{\omega}\sin\omega t\right]_0^{\frac{2\pi}{\omega}}$$

$$< \cos\omega t >_T = \left|\frac{1}{\omega}\sin\omega t\right|_0^{\frac{2\pi}{\omega}}$$

$$< \cos\omega t >_T = \left[\frac{1}{\omega}\sin\omega\left(\frac{2\pi}{\omega}\right) - \frac{1}{\omega}\sin\omega(0)\right]$$

$$< \cos\omega t >_T = 0$$

Similarly;

$$< \sin\omega t >_T = 0$$

$$< \sin^2\omega t >_T = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \sin^2\omega t\, dt$$

From; $\cos 2\omega t = 1 - 2\sin^2\omega t$

$$\sin^2\omega t = \frac{1}{2}(1 - \cos 2\,\omega t)$$

$$< \sin^2\omega t >_T = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{1}{2}(1 - \cos 2\,\omega t)dt$$

$$< \sin^2\omega t >_T = \frac{\omega}{4\pi}\left[\int_0^{\frac{2\pi}{\omega}} dt - \int_0^{\frac{2\pi}{\omega}} \frac{1}{2}(\cos 2\,\omega t)dt\right]$$

$$< \sin^2\omega t >_T = \frac{\omega}{4\pi}\left[|t|_0^{\frac{2\pi}{\omega}} - \left|\frac{1}{2\omega}(\sin 2\,\omega t)\right|_0^{\frac{2\pi}{\omega}}\right]$$

$$< \sin^2\omega t >_T = \frac{\omega}{4\pi}\left[\left(\frac{2\pi}{\omega} - 0\right) - \left(\frac{1}{2\omega} \times \sin 2\,\omega \frac{2\pi}{\times\,\omega} - 0\right)\right]$$

$$< \sin^2\omega t >_T = \frac{1}{2}$$

Similarly;

$$< \cos^2\omega t >_T = \frac{1}{2}$$

**5. Root mean square value of alternating current or voltage,** $I_{r.m.s}$ and $V_{r.m.s}$

The root mean square value of alternating current is the value of steady current which dissipates electrical energy in a resistor at the same rate as the alternating current.

The root mean square value of alternating voltage is the value of steady voltage which dissipates electrical energy in a resistor at the same rate as the alternating voltage.

The root mean square value of alternating current is also the square root of the mean value of the square of current taken over a complete cycle.

$$\text{Root Mean Square value} = \sqrt{< [f(t)]^2 >_T}$$

$$I_{r.m.s} = \sqrt{< [I]^2 >_T}$$

$$I_{r.m.s} = \sqrt{< {I_0}^2 \sin^2\omega t >_T}$$

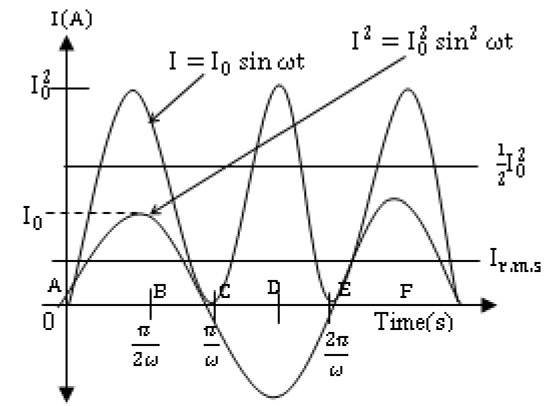
$$I_{r.m.s} = I_0\sqrt{< \sin^2\omega t >_T}$$

$$I_{r.m.s} = I_0\sqrt{\left(\frac{1}{2}\right)}$$

$$\mathbf{I_{r.m.s}} = \frac{\mathbf{I_0}}{\sqrt{2}}$$

Similarly

$$\mathbf{V_{r.m.s}} = \frac{\mathbf{I_0}}{\sqrt{2}}$$



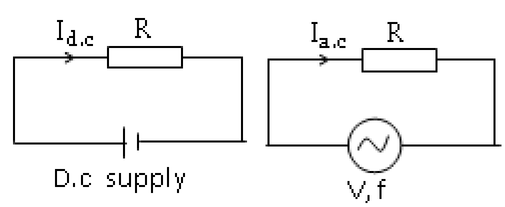

Note: For all calculations involving alternating currents, voltages, energy and power, root mean square values should be used.

Root mean square values should be used because, by using them, many calculations can be done as they would be for direct current.

**Relationship between peak current and** Root **mean square value of current**.

Both a.c and d.c currents have a heating effect through the $I^2R$mechanism. There is a value of d.c which causes the same (equal) heating effect in a resistor as the a.c. This value is called the root mean square value of a.c.

Consider a resistor in series with a.c source of electrical energy.




The instantaneous current I is given by

$$I = I_0 \sin\omega t \quad \text{or} \quad I = \frac{V_0}{R}\sin\omega t$$

$I_0 = \frac{V_0}{R}$

The instantaneous power dissipated in the resistor,

$P = I^2R = (I_0 \sin \omega t)^2 R = I_0^2 R \sin^2 \omega t$

For the root mean square value;

$P_{d.c} = P_{a.c}$

$I_{d.c}^2 R = I_{a.c}^2 R$

$I_{d.c}^2 R = (I_0 \sin \omega t)^2 R$

$\quad = {I_0}^2 R \sin^2 \omega t$

The average power over one cycle, $< P_{a.c} >_T$

$< P_{d.c} >_T = < P_{a.c} >_T$

$< I_{d.c}^2 R > = {I_0}^2 R < \sin^2 \omega t >_T$

$< I_{d.c}^2 > = {I_0}^2 < \sin^2 \omega t >_T$

$< I_{d.c}^2 > = {I_0}^2 \left(\frac{1}{2}\right)$

But $\mathbf{I_{r.m.s}}$ is the value of steady current which dissipates electrical energy in the resistor at the same rate as a.c. thus;

$I_{r.m.s}^2 = {I_0}^2 \left(\frac{1}{2}\right)$

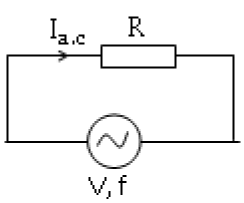
$\mathbf{I_{r.m.s}} = \frac{\mathbf{I_0}}{\sqrt{2}}$

Similarly

$\mathbf{V_{r.m.s}} = \frac{\mathbf{I_0}}{\sqrt{2}}$

**Resistor in a.c circuits**

Consider a resistor in series with a.c source of electrical energy.




The instantaneous current I is given by

$I = I_0 \sin \omega t$ or $I = \frac{V_0}{R} \sin \omega t \ldots\ldots\ldots.(i)$

From Ohm's law;

$V = IR$

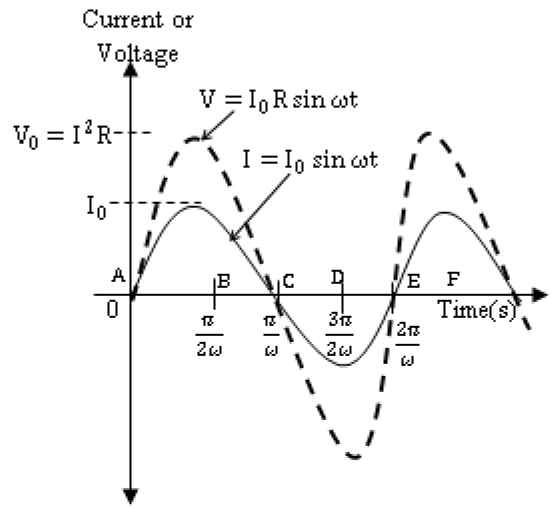
$V = (I_0 \sin \omega t)R$

$V = I_0 R \sin \omega t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$

Hence for a resistor in series with a.c source of electrical energy, V and I are in phase. That is to say they both reach the maximum and minimum values at the same time.

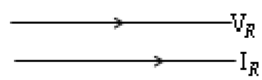
Thus the circuit above is called a non reactive circuit.

A sketch graph of I and V against time, t for a.c through a pure resistor.




The peak value of voltage is greater than the peak value of current.

Phasor or vector diagram



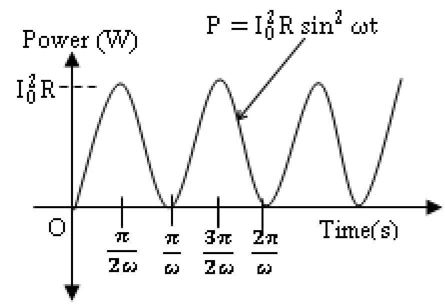

Power dissipated in a resistor.

-When current passes through a resistor of resistance, R, power is expended to overcome the resistance.

-The instantaneous power dissipated in the resistor, is given by;

$P = I^2R = (I_0 \sin \omega t)^2 R = I_0^2 R \sin^2 \omega t$

Graph of power in a resistor against time




Thus the mean power dissipated over a full cycle is given by;

$< P >_T = < I_0^2 R \sin^2 \omega t >_T$

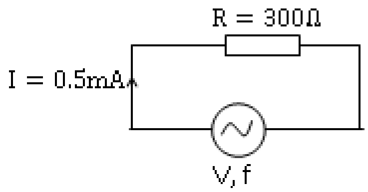
$< P >_T = I_0^2 R < \sin^2 \omega t >_T$

$< \boldsymbol{P} >_T = \boldsymbol{I_0^2 R} \left(\frac{1}{2}\right) = \frac{\boldsymbol{I_0^2 R}}{2}$

**Examples:**

1. The peak value of an alternating current through a 300Ω resistor is 0.5mA. Calculate the power dissipated by the resistor.

**Solution:**




$I_0 = 0.5\text{mA} = 0.5 \times 10^{-3}\text{A}; \quad R = 300\Omega$

$I_{r.m.s} = \frac{I_0}{\sqrt{2}}$

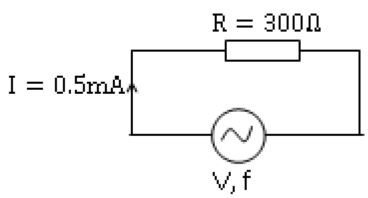
$\text{Power} = I_{r.m.s}^2 R = \left(\frac{I_0}{\sqrt{2}}\right)^2 R$

$\text{Power} = \left(\frac{0.5 \times 10^{-3}}{\sqrt{2}}\right)^2 \times 300$

$\text{Power} = 3.75 \times 10^{-5}\text{W}$

2. An alternating current $I = 0.8sin120ðt\ A$ flows through a resistance of 16Ω. Find the mean power dissipated. Hence deduce the r.m.s value of the alternating current.

**Solution:**




$I = 0.8sin120ðt\ A; \quad R = 16\Omega$

$< P >_T = I_0^2 R < \sin^2 ùt >_T$

$< P >_T = < 0.8^2(16) < \sin^2 120ðt >_T$

$But\ < \sin^2 120ðt >_T = \frac{1}{2}$

$< P >_T = 0.8^2(16)\left(\frac{1}{2}\right);$

$< P >_T = 5.12\text{ W}$

From definition of r.m.s value of current as the steady value of d.c that dissipates heat at the same rate as a.c.

The average power over one cycle, $< P_{a.c} >_T$

$< P_{d.c} >_T = < P_{a.c} >_T$

$< I_{d.c}^2 R > = 5.12$

$< 16 I_{d.c}^2 > = 5.12$

$< I_{d.c}^2 > = \frac{5.12}{16}$

$I_{r.m.s}^2 = \frac{5.12}{16}$

$I_{r.m.s} = 0.566\text{A}$

<u>Alternatively;</u>

$I_{r.m.s} = \frac{I_0}{\sqrt{2}}$

$\text{Power} = I_{r.m.s}^2 R = \left(\frac{I_0}{\sqrt{2}}\right)^2 R$

$\text{Power} = \left(\frac{0.8}{\sqrt{2}}\right)^2 \times 16$

$\text{Power} = 5.12\text{ W}$

3. An alternating current $I = 0.4sin120ðt\ A$ flows through a resistance of 16Ω. Find the ;

(i) Frequency of the a.c.
(ii) Peak value of the current
(iii) R.m.s values of the current and voltage
(iv) Power dissipated

**Solution:**

**(i)**

$I = 0.4\sin 120\pi t\text{ A}; \quad R = 16\Omega$

Compare with the sinusoidal current

$I = I_0 \sin \omega t$

$I = I_0 \sin 2\pi f t$

Thus: $2\pi f t = 120\pi t$

$f = 60\text{Hz}$

**(ii)**

$I_{r.m.s} = \frac{I_0}{\sqrt{2}} = \frac{0.4}{\sqrt{2}} = 0.283\text{ A}$

$V_{r.m.s} = I_{r.m.s}R = 0.283(16) = 4.53\text{V}$

**(iii)**

$\text{Power} = I_{r.m.s}^2 R$

$= (0.283)^2 \times 16$

$\text{Power} = 1.28\text{ W}$

4. An alternating voltage $V = 120\sin 120\pi f t$ volts is applied across a kettle of resistance 20Ω containing 800g of water at $25^0 C$. How long will it take the temperature of the water ti rise to $80^0 C$.

**Solution:**

$V_{r.m.s} = \frac{V_0}{\sqrt{2}} = \frac{120}{\sqrt{2}}\text{Volts} \quad ; R = 20\Omega$

Assuming no heat losses,;

Heat energy lost by the kettle

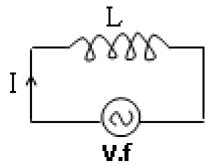
= Heat energy gained by water

$\frac{V^2 t}{R} = mc\Delta\theta$

$\frac{V_{r.m.s}^2 t}{R} = mc\Delta\theta$

$$\frac{\left(\frac{120}{\sqrt{2}}\right)^2 t}{20} = 0.8 \times 4200(80 - 25)$$
$$t = 513.3\ s$$

**Inductor in a.c circuits:**
A pure inductor is a coil of negligible resistance. .




Assuming sinusoidal current I is passed through the inductor.
$I = I_0 \sin \omega t$ … … … … … … … … … … … … … … … (i)

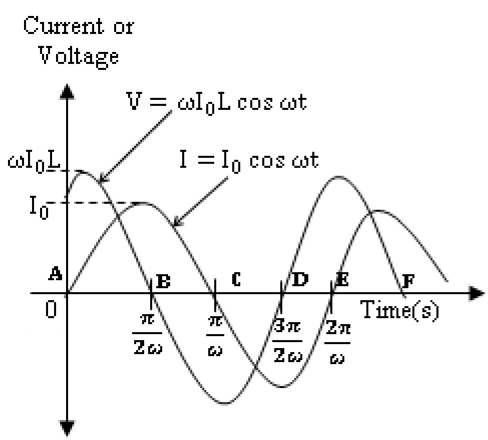
A.c flows through the inductor, L since is made up of coils. The changing current sets up a back e.m.f in the coil which is given by;

$$E_b = -L\frac{dI}{dt} = -LI_0\frac{d(\sin \omega t)}{dt} = -\omega L I_0 \cos \omega t$$

Assuming the inductor has zero resistance, then for current to flow, the applied p.d,V must be equal and opposite to the back emf, hence ;
$V = -E_b$
$V = \omega L I_0 \cos \omega t = V_0 \cos \omega t$ … … … … … … (ii)
$\mathbf{V} = \omega L I_0\ \mathbf{sin}\left(\omega \mathbf{t} + \frac{\pi}{2}\right) = \mathbf{V}_0 \mathbf{sin}\left(\omega \mathbf{t} + \frac{\pi}{2}\right)$
Where $V_0 = \omega L I_0$

Thus, for a.c through a inductor, voltage and current have the same frequency but are out of phase by $\frac{\pi}{2}$ radians 0r $90^0$ and voltage leads current by $\frac{\pi}{2}$ radians 0r $90^0$.
Therefore a circuit with an inductor is a reactive circuit.

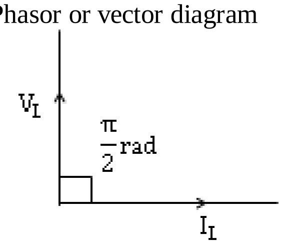
<u>A sketch graph of I and V against time, t for a.c through an inductor.</u>

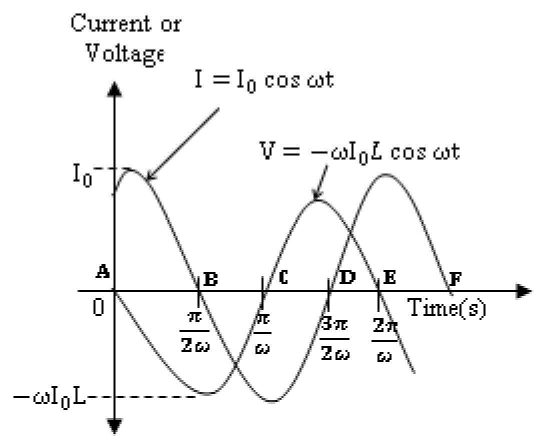



The peak value of voltage is greater than the peak value of current.

**Explanation:**
-The back e.m.f is induced in the coil as a result of the change in current flowing in it. Its magnitude is directly proportional to the rate of change of current and is given by; $E_b = -L\frac{dI}{dt}$.
-The rate of change of current $\left(V = -E_b = L\frac{dI}{dt}\right)$ decreases with increasing current and increases with decreasing current.
-At A, when $I = 0, \frac{dI}{dt}$ is maximum and hence voltage is maximum. During the next quarter cycle, the voltage increases in the opposite direction with decreasing current (since $\frac{dI}{dt}$ is negative) until it becomes maximum at C, when current is zero again.
-The process is repeated during the next two quarters of the cycle and thus the voltage leads the current by a phase angle of $\frac{\pi}{2}$ radians.
Note: For a pure inductor, the back e.m.f is equal to the applied voltage.

Phasor or vector diagram




**Explanation:**
-The back e.m.f is induced in the coil as a result of the change in current flowing in it. Its magnitude is directly proportional to the rate of change of current and is given by; $E_b = -L\frac{dI}{dt}$.
-The rate of change of current $\left(V = -E_b = L\frac{dI}{dt}\right)$ decreases with increasing current and increases with decreasing current.
-At A, when $t = 0, I = \max, \frac{dI}{dt} = 0$ and hence voltage is zero$V = 0$.

-During the first quarter cycle, the current decreases and $\frac{dI}{dt}$ increases with decreasing current until it becomes maximum at B, when current is zero. (V is negative since $\frac{dI}{dt}$ is negative)
-During the next quarter cycle, the current increases in the opposite direction and $\frac{dI}{dt}$ decreases with increasing current and hence the voltage decreases until it becomes zero$\left(\frac{dI}{dt} = 0\right)$ at C when the current is maximum.
-The process is repeated during the next two quarters of the cycle and thus the voltage leads the current by a phase angle of $\frac{ð}{2}$ radians.
**Note:** For a pure inductor, the back e.m.f is equal to the applied voltage.

**Alternatively:**
Assuming that current is sinusoidal,
$V = V_0 \cos \omega t$

$V = -E_b$
$V = -\left(-L\frac{dI}{dt}\right) = L\frac{dI}{dt}$
$Vdt = LdI$
$(V_0 \cos \omega t)dt = LdI$

$\frac{V_0}{L}\int_0^t (\cos \omega t)dt = \int_0^I dI$
$\frac{V_0}{L}\left[\frac{1}{ù} \times (\sin \omega t)\right]_0^t = [I]_0^I$
$\frac{V_0}{\omega L}\sin \omega t = I$
$\mathbf{I} = \frac{V_0}{\omega L}\sin \omega t = I_0\ \mathbf{sin}\ ù\mathbf{t}$

$I = \frac{V_0}{\omega L}\cos\left(\omega t + \frac{\pi}{2}\right) = I_0 \cos\left(\omega t + \frac{\pi}{2}\right)$

Where, $I_0 = \frac{V_0}{\omega L} = \text{Peak value of current}$

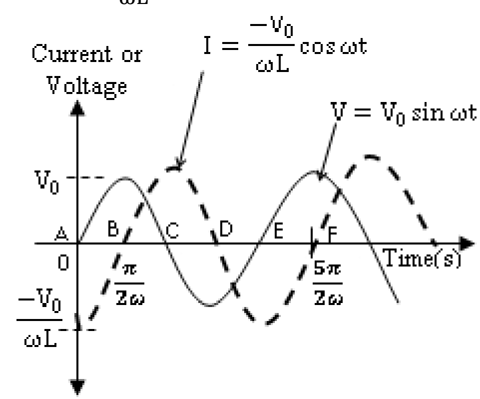
**Alternatively:**
Assuming that voltage is sinusoidal,
$V = V_0 \sin \omega t$

$V = -E_b$
$V = -\left(-L\frac{dI}{dt}\right) = L\frac{dI}{dt}$
$Vdt = LdI$
$(V_0 \sin \omega t)dt = LdI$

$\frac{V_0}{L}\int_0^t (\sin \omega t)dt = \int_0^I dI$
$\frac{V_0}{L}\left[\frac{1}{ù} \times (-\cos \omega t)\right]_0^t = [I]_0^I$
$\frac{-V_0}{\omega L}\cos \omega t = I$
$\mathbf{I} = \frac{-V_0}{\omega L}\cos \omega t = -I_0\ \mathbf{cos}\ ù\mathbf{t}$

$I = \frac{-V_0}{\omega L}\sin\left(\omega t + \frac{\pi}{2}\right) = -I_0 \sin\left(\omega t + \frac{\pi}{2}\right)$

Where, $I_0 = \frac{V_0}{\omega L} = \text{Peak value of current}$




**Inductive reactance, $X_L$ :**
Inductive reactance is the opposition to flow of a.c in an inductor. It is measured in Ohms, (Ω).
From Ohm's law;
$V_{r.m.s} = I_{r.m.s}X_L$

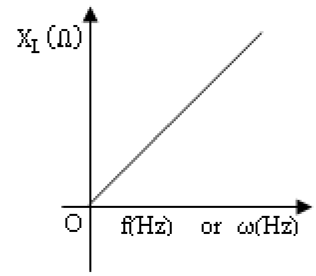
$X_L = \frac{V_{r.m.s}}{I_{r.m.s}} = \frac{\left(\frac{V_0}{\sqrt{2}}\right)}{\left(\frac{I_0}{\sqrt{2}}\right)} = \frac{V_0}{\sqrt{2}} \times \frac{\sqrt{2}}{I_0} = \frac{V_0}{I_0}$

$\text{Inductive reactance} = \frac{\text{Peak voltage}}{\text{Peak current}}$

Therefore, from;
$V = \omega L I_0 \cos \omega t = V_0 \cos \omega t$

$X_L = \frac{V_0}{I_0} = \frac{\omega L I_0}{I_0} = \omega L = 2\pi f L$
Where f is the frequency of the alternating current.

$$\text{Thus; } X_L \propto L \text{ and } X_L \propto f$$
Therefore in a capacitor, $X_L$ has a linear relationship with ω or f:



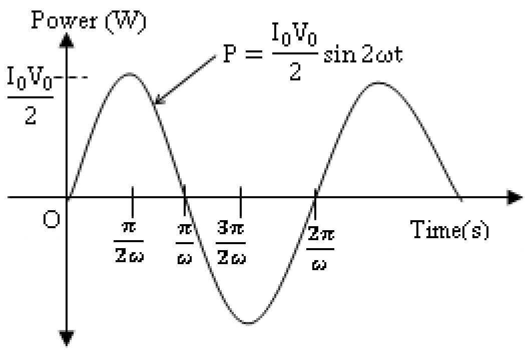

Instantaneous power developed in inductive circuits:

$P = IV = (I_0 \sin \omega t)(V_0 \cos \omega t)$

$P = I_0 V_0 (\sin \omega t \cos \omega t)$

$P = \frac{I_0 V_0}{2}(\sin 2\omega t)$

Thus theFrequency of power is double the frequency of voltage.




Thus the mean or average power dissipated over a full cycle is given by;

$< P >_T = < \frac{I_0 V_0}{2}(\sin 2\omega t) >_T$

$< P >_T = \frac{I_0 V_0}{2} < \sin 2\omega t >_T$

$< P >_T = \frac{I_0 V_0}{2}(0) = 0$

$< \mathbf{P} >_{\mathbf{T}} = 0$

There fore, the average Power over one cycle is zero. Hence n inductor is a wattles component in a.c circuits.

**Why average power is zero**

-When the current rises, the back emf opposes the flow of current. The current flows against the back emf and therefore does work against it.

-The total work done in bringing the current to its final value is stored in the magnetic field of the coil. It is liberated when the current collapses, therefore the back emf tends to maintain the current and do external work.

-Hence, during the first quarter cycle, the magnetic field linking the coil is building up to the maximum value. Energy is supplied by the source and stored by the magnetic filed of the coil.

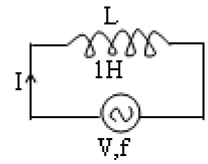
-During the second quarter cycle, the current in the coil decreases to zero. The energy that was stored in the coil is returned to the source.

-In the third quarter cycle, the source supplies current in the reverse direction. The energy supplied by the source is stored in the magnetic field of the coil and in the 4th cycle an equal amount of energy is restored to the store.

-Hence the energy dissipated over one cycle in a circuit is zero.

**Examples:**

An ideal inductor of inductance 1H is connected across an a.c source as shown below.




If the current at any instant is given by

$I = 0.03 \sin 100\pi t.$

(i) Determine the r.m.s value of the applied voltage

(ii) What is the power dissipated in the circuit.

(iii) Qualitatively account for your results.

**Solution:**

**(i)**

$X_L = \omega L = 100\pi(1) = 100\pi\ \Omega$

$V_0 = I_0 X_L$

$V_0 = 0.03(100\pi) = 3\pi$

$V_{r.m.s} = \frac{V_0}{\sqrt{2}} = \frac{3\pi}{\sqrt{2}} = 6.664$ A

(ii)

$< P >_T = \frac{I_0 V_0}{2} < \sin 2\omega t >_T$

$< P >_T = \frac{I_0 V_0}{2}(0) = 0$

$< P >_T = 0$

2.

(a) An alternating voltage, $V = V_0 \cos 2\pi f t$ volts was applied across a coil of self-inductance, L henrys.

(i) Find the expression for the current flowing.

(ii) Using the same axes, sketch graphs of current and voltage against time.

(b) A coil of self- inductive 2H having negligible resistance is connected to an a.c circuit where a current, $I = 0.04 \sin 100\pi t$ amperes flows. Find the;

(i) Reactance of the coil. [$X_L = 628.32\ \Omega$ ]

(ii) r.m.s value of the p.d across the coil.

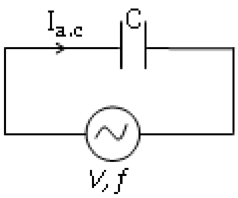
[$V_{r.m.s} = 17.77$ A]

(c) An alternating voltage, $V = V_0 \sin \omega t$ volts was applied across a coil of self-inductance, L henrys and having negligible resistance.

(i) Find the expression for the current flowing.
(ii) Using the same axes, sketch graphs of current and voltage against time.

(d) A coil of self inductance 0.8H is connected in an a.c circuit of 240V, 50Hz. Find the;
(i) r.m.s value of the current flowing.
$[I_{r.m.s} = 0.9554\ \mathrm{A}]$
(ii) Power dissipated in the circuit
$[P = 0\ \mathrm{W}]$
**Note**: **240 V** is a **a r.m.s** voltage

**Capacitors in a.c circuits:**
Consider a capacitor of capacitance C, connected in series with an a.c source.




Assuming that voltage is sinusoidal,
$V = V_0 \sin \omega t$

Hence charge Q on the plates of the capacitor is;
$Q = CV = CV_0 \sin \omega t$

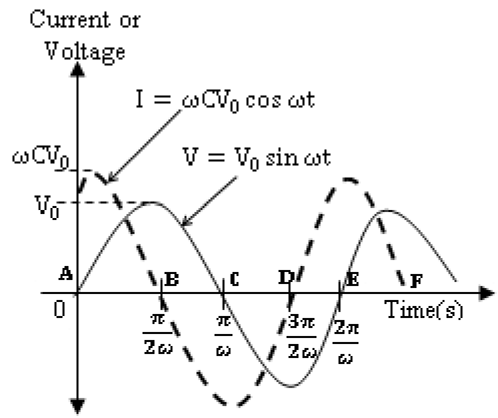
The current at any time t is equal to the rate of change of charge.

$$I = \frac{dQ}{dt} = \frac{d(CV_0 \sin \omega t)}{dt} = CV_0 \frac{d(\sin \omega t)}{dt}$$
$$\mathbf{I = \omega CV_0 \cos \omega t = I_0 \cos \omega t}$$

$$\mathbf{I = \omega CV_0 \sin\left(\omega t + \frac{\pi}{2}\right) = I_0 \sin\left(\omega t + \frac{\pi}{2}\right)}$$
Where, $I_0 = \omega CV_0 = \text{Peak value of current}$

Thus, for a.c through a capacitor, current and voltage are out of phase by $\frac{\pi}{2}\text{radians}\ \ 0\text{r}\ 90^0$ and current leads voltage by $\frac{\pi}{2}\text{radians}\ \ 0\text{r}\ 90^0$.
Therefore a circuit with a capacitor is a reactive circuit.

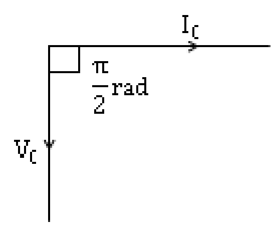
<u>A sketch graph of I and V against time, t for a.c through a capacitor.</u>




The peak value of current is greater than the peak value of voltage.

**Explanation:**
-The charge on the capacitor plates varies directly with voltage. Since current is the rate of flow of charge.; $I = \frac{dQ}{dt}$.
-At A, when $V = 0$, and hence, $Q = 0$, $\left(I = \frac{dQ}{dt}\right)$ is maximum.
-During the first quarter cycle, the voltage increases and the capacitor charges. $\left(I = \frac{dQ}{dt}\right)$ decreases with increasing voltage and hence the charge on the capacitor plates decreases until it is zero at B, when the charge (or voltage) is maximum.
-During the next quarter of the cycle, the voltage decreases and the capacitor discharges sending current in the opposite direction. The current increase with decreasing P.d across the plates until it becomes maximum at C when the P.d is zero.
-The process is repeated during the next two quarters of the cycle and thus the current leads the voltage by a phase angle of $\frac{\pi}{2}$ radians.

Phasor or vector diagram




**Alternatively:**
Assuming that current is sinusoidal,
$I = I_0 \sin \omega t$

Hence charge Q on the plates of the capacitor is;
$Q = CV = CV_0 \sin \omega t$

The current at any time t is equal to the rate of change of charge.
$$I = \frac{dQ}{dt}$$

$$\frac{dQ}{dt} = I$$

$$dQ = Idt$$
$$d(CV) = Idt$$
$$CdV = Idt$$

$$C\int_0^V dV = I_0\int_0^t (\sin\omega t)dt$$
$$C[V]_0^V = I_0\left[\frac{1}{ù}\times(-\cos\omega t)\right]_0^t$$
$$CV = -\frac{I_0}{ù}\cos\omega t$$
$$\mathbf{V = -\frac{I_0}{ùC}\cos ùt = -V_0\cos ùt}$$

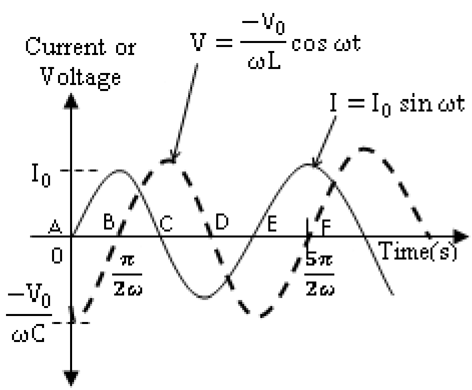
$$V = -\frac{I_0}{ùC}\sin\left(\omega t + \frac{\pi}{2}\right) = -V_0\sin\left(\omega t + \frac{\pi}{2}\right)$$

$$\mathbf{V = \frac{I_0}{ùC}\sin\left(ùt - \frac{ð}{2}\right) = V_0\sin\left(ùt - \frac{ð}{2}\right)}$$

Where, $V_0 = \frac{I_0}{ùC} = \text{Peak value of voltage}$

Assuming that current is sinusoidal,
$I = I_0 \sin\omega t$

<u>A sketch graph of I and V against time, t for a.c through a capacitor.</u>




**Explanation:**

-The charge on the capacitor plates varies directly with voltage. Since current is the rate of flow of charge.; $I = \frac{dQ}{dt}$.

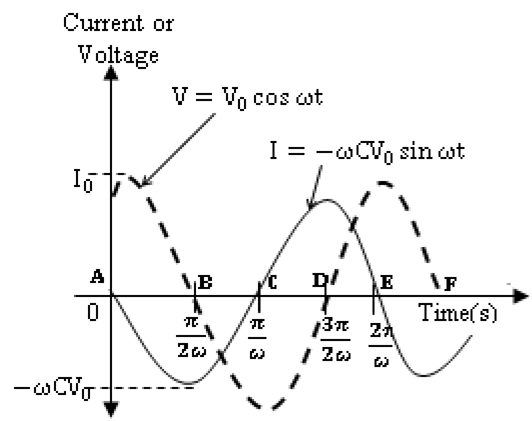
-At A, when $V = \text{maximum, and hence,}$ the charge on the capacitor plates, $Q = \text{maximum,}$
$\text{The current or rate of flow of charge}$ $\left(I = \frac{dQ}{dt}\right)$ is zero.

-During the first quarter cycle, the voltage decreases and the capacitor discharges. The rate of charge flow (current) i.e $\left(I = \frac{dQ}{dt}\right)$ increases with increasing voltage (or charge) until it is maximum at B, when the charge (or voltage) is zero.

-During the next quarter of the cycle, the voltage increases and the capacitor charges. The current decreases with increasing voltage or charge until it reduces to zero at C when the voltage (or charge) is maximum

-The process is repeated during the next two quarters of the cycle and thus the current leads the voltage by a phase angle of $\frac{ð}{2}$ radians.

Assuming that voltage is a cosine function;
$V = V_0 \cos\omega t$




**Capacitive reactance, $X_c$ :**

Capacitive reactance is the opposition to flow of a.c in a capacitor. It is measured in Ohms, (Ω).

From Ohm's law;
$V_{r.m.s} = I_{r.m.s}X_c$

$$X_c = \frac{V_{r.m.s}}{I_{r.m.s}} = \frac{\left(\frac{V_0}{\sqrt{2}}\right)}{\left(\frac{I_0}{\sqrt{2}}\right)} = \frac{V_0}{\sqrt{2}}\times\frac{\sqrt{2}}{I_0} = \frac{V_0}{I_0}$$

$$\text{Capacitive reactance} = \frac{\text{Peak voltage}}{\text{Peak current}}$$

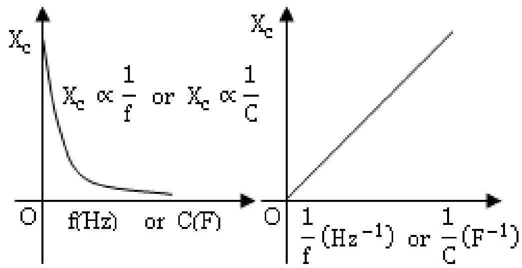
Therefore, from;
$V = V_0 \sin\omega t \text{ and} I = \omega\, CV_0 \cos\omega t$

$$X_c = \frac{V_0}{I_0} = \frac{V_0}{\omega CV_0} = \frac{1}{\omega C} = \frac{1}{2\pi fC}$$

Where f is the frequency of the alternating current.

$$\text{Thus; } X_c \propto \frac{1}{C} \quad \text{and} \quad X_c \propto \frac{1}{f}$$

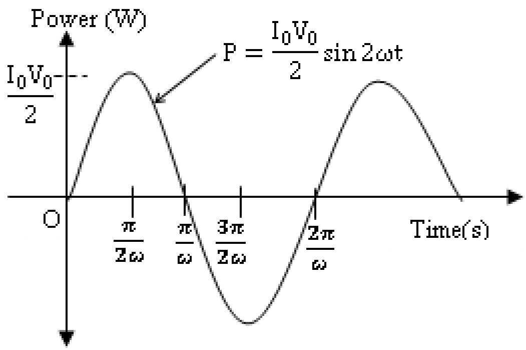
A graph of $X_c$ against frequency, f and that of $X_c$ against frequency, 1/f.




Instantaneous power developed in the capacitor,

$$P = IV = (I_0 \sin \omega t)(V_0 \cos \omega t)$$
$$P = I_0V_0(\sin \omega t \cos \omega t)$$
$$P = \frac{I_0V_0}{2}(\sin 2\,\omega t)$$

Thus theFrequency of power is double the frequency of voltage.




Thus the mean or average power dissipated over a full cycle is given by;

$$< P >_T = < \frac{I_0V_0}{2}(\sin 2\,\omega t) >_T$$
$$< P >_T = \frac{I_0V_0}{2} < \sin 2\,\omega t >_T$$
$$< P >_T = \frac{I_0V_0}{2}(0) = 0$$
$$< \boldsymbol{P} >_{\boldsymbol{T}} = 0$$

There fore, the average Power over one cycle is zero. Hence a capacitor is a wattles component in a.c circuits.

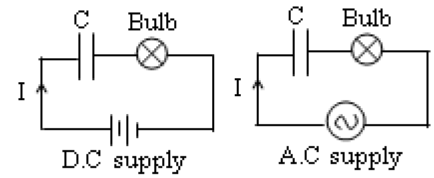
Reason why a capacitor is a wattless component

-During the first quarter cycle, the capacitor charges and energy is drawn from the source and stored in the electric field of the capacitor.

-During the second quarter cycle, the capacitor discharges and energy is returned to the source.

-During the third cycle, the capacitor charges in the opposite direction, again energy is stored in the electric field in the capacitor and in the last quarter, the capacitor discharges and energy returns to the source. Therefore in one cycle, there is no net energy stored in the capacitor.

**Question: Explain why a capacitor allows the flow of a.c but not d.c.**




**Observation:**

-When a capacitor is connected to a d.c source, the bulb does not light up.

-When a capacitor is connected to an a.c source, the bulb lights up.

**Explanation:**

-A capacitor is two plates with an insulator between them. Hence, it does not pass d.c neither a.c through it. It blocks both!

-When a voltage is apply across the electrodes of a capacitor, the dipoles present in the dielectric media get polarised and a form of displacement current is established in the circuit.

-Even when a DC voltage is applied to a capacitor which is not charged a current will flow till the capacitor is fully charged as in the process of charging there exist . Once it is fully charged no additional charge is pumped in or out of the capacitor and it blocks current.

-In case of a.c, because of change in polarity of the applied voltage there will be continuous polarisation and depolarisation in each cycle causing rate of change of charge stored in the capacitor and hence flow of current with a 90 degree phase shift with the applied voltage.

-Hence, current is flowing, not in the capacitor, but rather in the loop as charging and discharging current cycle the circuit.

**Examples**

1. A capacitor of 1μC and capacitance 1F is used in a radio circuit where frequency is 1000Hz and current is 2mA. Calculate the voltage across C.

$$\text{Reactance}, X_c = \frac{V_0}{I_0} = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

$$X_c = \frac{1}{2\pi \times 1000 \times 1 \times 10^{-6}} = 159\Omega$$

From Ohm's law;

$$V = IX_c$$
$$V = (2 \times 10^{-3}) \times 159 = 0.32\text{ V}$$

2. A sinusoidal voltage of r.m.s 10V is supplied a cross a 50μF capacitor.

(i) Find the peak value of charge on the capacitor plates.

**Solution:**

$$Q = CV = CV_0 \sin \omega t = Q_0 \sin \omega t$$
$$\text{Where}, Q_0 = CV_0$$
$$Q_0 = C(I_{r.m.s}\sqrt{2})$$
$$Q_0 = (50 \times 10^{-6})(10)\sqrt{2}$$
$$Q_0 = 7.07 \times 10^{-4}C$$

(ii) If the a.c supply has a frequency of 50Hz, calculate the r.m.s value of current through the capacitor.

$$X_c = \frac{V_0}{I_0} = \frac{V_{r.m.s}}{I_{r.m.s}} = \frac{1}{2\pi fC}$$

$$I_{r.m.s} = V_{r.m.s}(2\pi fC)$$
$$= 10(2\pi)(50)(50 \times 10^{-6})$$
$$I_{r.m.s} = 1.57 \times 10^{-1}\text{A}$$

(iii). On the same axes, draw a graph of charge, Q and current I against time for the capacitor.

$$Q = CV_0 \sin \omega t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$$

$$I = \frac{dQ}{dt} = \frac{d(CV_0 \sin \omega t)}{dt} = \omega\, CV_0 \cos \omega t$$
$$I = \omega\, CV_0 \cos \omega t \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

$$I = \omega\, CV_0 \sin\left(\omega t + \frac{\pi}{2}\right)$$

Thus I leads Q by $\frac{ð}{2}$radians

[See graph of V and I against time for a capacitor.]

**Questions**

1. A sinusoidal a.c $I = 4\sin 100\pi t$Amperes flows through a resistor of resistance 2Ω. Find the mean power dissipated in a resistor and hence deduce the r.m.s value of current.
$[< P >_T = 16\text{W}, I_{r.m.s} = 2\sqrt{2}\text{A}]$

2. What is the r.m.s value of alternating current which must pass through a resistor immersed in oil in a calorimeter so that the initial rate of rise of temperature of the oil is three times that produced when a direct current of 2A passes through the resistor under the same conditions?[3.46A]

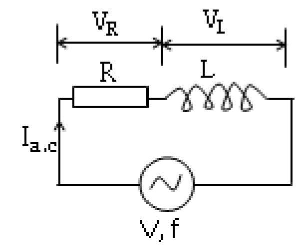
3. A pure inductor of self inductance 1H is connected across an alternating voltage of 115V and frequency 60Hz. Calculate the
(i) Inductive reactance (ii) inductive current
(iii) Peak current (iv) average power consumed. [377.1Ω, 0.3A, 0.43A, zero watts]

4. A 100V, 50Hz a.c. supply is connected across 24μF capacitor as shown below.
(i) Calculate the reactance of the circuit.
[Ans: 132.63Ω]
(ii) Sketch a graph to show the time dependence of the applied voltage and the current through the circuit.

5. An inductor of 2H and negligible resistance is connected to a 12V mains supply, frequency 50Hz. Find the current flowing. $[I = 159\text{ A}]$.

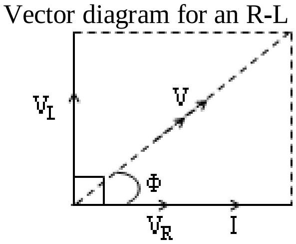
**SERIES CIRCUITS:**

1. **An inductor and Resistor in series**




-For a resistor, both current and voltage are in phase (they are maximum and minimum at the same time).
-For an inductor, voltage leads current by $\frac{ð}{2}$ radians

Vector diagram for an R-L circuit.




The supply voltage V can be obtained by Pythagoras theorem.

$$V^2 = {V_L}^2 + {V_R}^2$$
$$V^2 = (IX_L)^2 + (IR)^2$$
$$V^2 = I^2\left({X_L}^2 + R^2\right)$$
$$\mathbf{I} = \sqrt{\frac{V^2}{\left({X_L}^2 + R^2\right)}} = \frac{\mathbf{V}}{\sqrt{\left(\mathbf{X_L}^2 + \mathbf{R}^2\right)}}$$

**Impedance, Z:**
This is the total opposition to the flow of a.c through the components of a circuit where an inductor and resistor or capacitor and resistor are connected in series.

From Ohm's law:

$$V = IR \iff R = \frac{V}{I}$$

$$\text{Impedance}, Z = \frac{V}{I}$$

$$\text{Impedance}, Z_L = V \times \frac{\sqrt{\left(\mathbf{X_L}^2 + \mathbf{R}^2\right)}}{\mathbf{V}}$$

$$\textbf{Impedance}, \mathbf{Z_L} = \sqrt{\left(\mathbf{X_L}^2 + \mathbf{R}^2\right)}$$

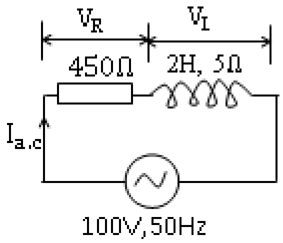
Phase angle: is the angle between the resultant voltage and the current flowing through the circuit.

$$\tan \Phi = \frac{V_L}{V_R} = \frac{IX_L}{IR} = \frac{\omega L}{R}$$

$$\mathbf{tan}\, \text{Ö} = \frac{\text{ù}\mathbf{L}}{\mathbf{R}} \Leftrightarrow \text{Ö} = \mathbf{tan}^{-1}\left(\frac{\text{ù}\mathbf{L}}{\mathbf{R}}\right)$$

-Thus the supply voltage leads current by $\frac{\eth}{2}$ radians

**Example 1:**




Find the current flowing in the circuit above.

**Solution:**

- Total resistance, $R = 450 + 5 = 455\Omega$
- Reactance, $X_L = \omega L = 2\pi fL$

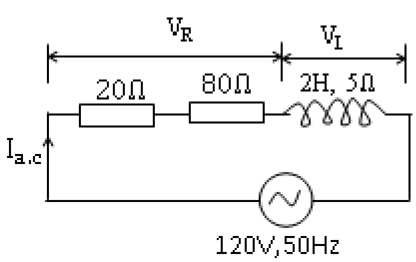
$$= 2\eth(50)(2)$$
$$= 200\eth\ \Omega$$

- Impedance, $Z_L = \sqrt{\left({X_L}^2 + R^2\right)}$

$\text{Impedance}, Z_L = \sqrt{(200\eth)^2 + (455)^2}$
$\text{Impedance}, Z_L = 803\Omega$
From Ohm's law;
$V = IZ$
$100 = I(803)$
$I = 0.125\ A$

**Example 2:**
A coil of self inductance 0.8H and having a resistance of 20Ω is connected in series with a resistance of 80Ω and an a.c mains of 120V.r.m.s, 50HZ. Find the:
(i) Peak value of current flowing.
(ii) r.m.s value of p.d across the the resistance.
(iii) Power dissipated in the coil.
(iv) Power dissipated in the resistor
(v) Phase angle between the applied voltage and the current.
(vi) Power factor

**Solution:**




(i)
$V = I\sqrt{\left({X_L}^2 + R^2\right)}$
$Z = \frac{V}{I} = \sqrt{(251.327^2 + (80 + 20)^2)}$
$Z = 270.49\Omega$

$V_{r.m.s} = I_{r.m.s}(Z)$
$120 = I_{r.m.s}(270.49)$
$I_{r.m.s} = 0.4436\ A$
$I_{r.m.s} = \frac{I_0}{\sqrt{2}} \Leftrightarrow I_0 = I_{r.m.s}\sqrt{2}$
$I_0 = (0.4436)\sqrt{2}$
$I_0 = 0.627\ A$

(ii)
$V_{r.m.s} = I_{r.m.s}X_L$
$V_{r.m.s} = 0.4436 \times 251.327$
$V_{r.m.s} = 111.49\ V$

(iii)
$V_{r.m.s} = I_{r.m.s}R$
$V_{r.m.s} = 0.4436 \times (80 + 20)$
$V_{r.m.s} = 44.36\ V$

$P = I_{r.m.s}^2 R$
$P = (0.4436)^2 \times 20$
$P = 3.936\ W$

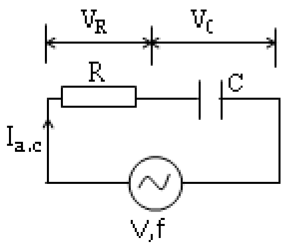
(iv)
$P = I_{r.m.s}^2 Z$
$P = (0.4436)^2 \times 270.49$
$P = 53.23\ W$

$P = I_{r.m.s}^2 R$
$P = (0.4436)^2 \times 80$
$P = 15.74\ W$

**(v)**
$\Phi = \tan^{-1}\left(\frac{X_L}{R}\right)$
$\Phi = \tan^{-1}\left(\frac{251.327}{20 + 80}\right)$
$Ö = 68.3^0$

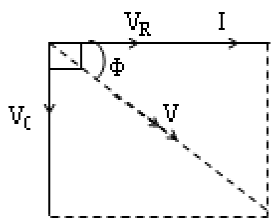
(vi)
Power factor $= \cos\Phi$
Power factor $= \cos 68.3^0$
**Power factor** $= 0.37$

2. **A capacitor and Resistor in series**




-For a resistor, both current and voltage are in phase (they are maximum and minimum at the same time).
-For a capacitor, current leads voltage by $\frac{\pi}{2}$ radians

Vector diagram for am R-C circuit.




The supply voltage V can be obtained by Pythagoras theorem.

$V^2 = V_C{}^2 + V_R{}^2$

$V^2 = (IX_C)^2 + (IR)^2$

$V^2 = I^2(X_C{}^2 + R^2)$

$$\mathbf{I} = \sqrt{\frac{V^2}{(X_C{}^2 + R^2)}} = \frac{\mathbf{V}}{\sqrt{(\mathbf{X_C}{}^2 + \mathbf{R}^2)}}$$

From Ohm's law:

$$V = IR \iff R = \frac{V}{I}$$

$$\text{Impedance}, Z = \frac{V}{I}$$

$$\text{Impedance}, Z_{capacitor} = V \times \frac{\sqrt{(X_C{}^2 + R^2)}}{V}$$

$$\textbf{Impedance}, \mathbf{Z}_{\boldsymbol{capacitor}} = \sqrt{(\mathbf{X_C}{}^2 + \mathbf{R}^2)}$$

Phase angle: is the angle between the resultant voltage and the current flowing through the circuit.

$$\tan\Phi = \frac{V_C}{V_R} = \frac{IX_C}{IR} = \frac{\left(\frac{1}{\omega C}\right)}{R} = \frac{1}{\omega RC} = \frac{1}{2\pi fRC}$$

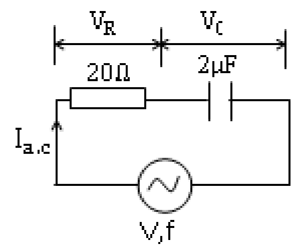
$$\mathbf{tan}\,\Phi = \frac{1}{\omega RC} \iff \Phi = \mathbf{tan}^{-1}\left(\frac{1}{\omega RC}\right)$$

**Examples:**

1. An alternating voltage $V = 120\sin 100\pi t$ volts is applied across a 2μF capacitor and a 20Ω resistor connected in series. Find the;

(i) Impedance of the circuit.
(ii) r.m.s value of the p.d across the capacitor.
(iii) Peak value of p.d across the resistor.
(iv) Power dissipated
(v) Phase angle between the applied voltage and current.

**Solution:**




- Total resistance, $R = 20\Omega$
- $V = 120\sin 100\pi t$

Compare with:

- $V = V_0 \sin \omega t$

Thus; $V_0 = 120$ V, $2\pi f = 100\pi \iff f = 50$Hz

- Reactance, $X_C = \frac{1}{\omega C} = \frac{1}{2\pi fC}$
- 

$$= \frac{1}{2\pi(50)(2\times10^{-6})}$$

$$= 1592.4\ \Omega$$

- Impedance, $Z = \sqrt{(X_C{}^2 + R^2)}$

Impedance, $Z = \sqrt{(1592.4)^2 + (20)^2}$

Impedance, $\mathbf{Z} = 1592.48\ \Omega$

From Ohm's law;

$V = IZ$

$V_{r.m.s} = I_{r.m.s}Z$

$$I_{r.m.s} = \frac{V_{r.m.s}}{Z} = \frac{\left(\frac{V_0}{\sqrt{2}}\right)}{Z} = \frac{\left(\frac{120}{\sqrt{2}}\right)}{1592.48} = 0.0533\text{ A}$$

$\mathbf{I_{r.m.s}} = 0.0533$ **A**

$V_{r.m.s} = I_{r.m.s}X_c$

$V_{r.m.s} = 0.0533(1592.4)$

$\mathbf{V_{r.m.s}}$ **=84.87 V**

$V_0 = I_0 R$

$$I_{r.m.s} = \frac{I_0}{\sqrt{2}} \iff I_0 = I_{r.m.s}\sqrt{2} = 0.0533(\sqrt{2})$$

$I_0 = 0.0754$ A

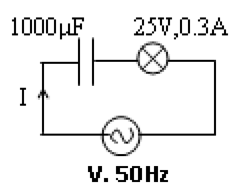
$V_0 = I_0 R$
$V_0 = 0.0754(20)$
$\mathbf{V_0 = 1.51\ V}$

(iv)**Note**: Power is dissipated in the resistive part of the circuit only.
$P = I^2_{r.m.s} R$
$P = 0.0533^2 \times 20$
$\mathbf{P} = 0.057\ \mathbf{W}$

$\Phi = \tan^{-1}\left(\frac{X_C}{R}\right)$
$\Phi = \tan^{-1}\left(\frac{1592.4}{20}\right)$
$\text{Ö} = 89.3^0$

**Example:2**
A 100µF capacitor is joined in series with a 2.5V, 0.30A lamp and a 50Hz supply. Calculate the
(i) P.d of the supply needed to light the lamp to its normal brightness.
(ii)P.d across the capacitor .

**Solution:**




a) Reactance of the capacitor, $X_c$:
$X_c = \frac{1}{2\pi fC} = \frac{1}{2\pi(50)(1000 \times 10^{-6})}$
$X_c = \frac{10}{\pi}\Omega$
Resistance of the lamp.
$V = IR \Leftrightarrow R = \frac{V}{I}$
$R = \frac{2.5}{0.3}$
$R = 8.3\ \Omega$

Impedance, Z of the circuit
$\text{Impedance}, Z = \sqrt{\left({X_C}^2 + R^2\right)}$
$\text{Impedance}, Z = \sqrt{\left(\frac{10}{ð}\right)^2 + (8.3)^2}$
$\text{Impedance}, Z = 8.9\ \Omega$
Then:

From Ohm's law; $V = IZ$
$V = 0.3(8.9)$
$V = 2.7V$

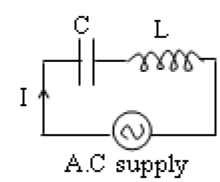
The phase angle between the applied P.d and current is given by;
$\Phi = \tan^{-1}\left(\frac{X_C}{Z}\right)$
$\Phi = \tan^{-1}\left[\frac{\left(10/ð\right)}{8.9}\right] =$

b) P.d across the capacitor.
From Ohm's law; $V = IX_C$
$V = 0.3\left(\frac{10}{\pi}\right)$
$V = 0.96\ V$

Note: $(V_R + V_C) = (2.5 + 0.96) = 3.5V$ This is greater than the supply voltage of 2.7V. This is because $V_R$ and $V_C$ are not in phase. Actually,

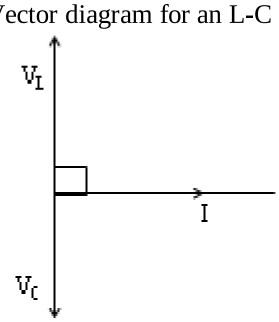
$$V^2 = {V_C}^2 + {V_R}^2$$

**3. An inductor and capacitor in series**



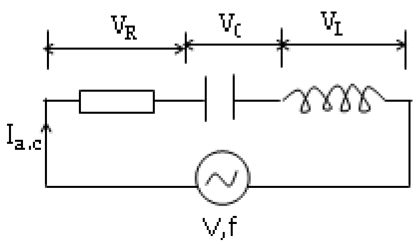

-For an inductor, voltage leads current by $\frac{ð}{2}$ radians
-For a capacitor, current leads voltage by $\frac{ð}{2}$ radians

Vector diagram for an L-C circuit.

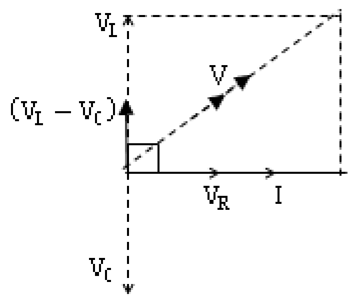



$V = V_L - V_C$
$Z = X_L - X_C$

**4. An inductor , capacitor and Resistor in series**




-For a resistor, current and voltage are in phase.
-For an inductor, voltage leads current by $\frac{ð}{2}$ radians
-For a capacitor, current leads voltage by $\frac{ð}{2}$ radians
Vector diagram for an R-C-L circuit.




-For a purely capacitive circuit,$V_C > V_L$
-For a purely inductive circuit,$V_L > V_C$.Therefore, the resultant vertical component of voltage is in the direction of the p.d across the inductor.

The supply voltage V can be obtained by Pythagoras theorem.

$$V^2 = (V_L - V_C)^2 + {V_R}^2$$
$$V^2 = (IX_L - IX_C)^2 + (IR)^2$$
$$V^2 = I^2[(X_L - X_C)^2 + (R)^2]$$
$$I = \sqrt{\frac{V^2}{[(X_L - X_C)^2 + (R)^2]}}$$
$$\mathbf{I} = \frac{\mathbf{V}}{\sqrt{(X_L - X_C)^2 + (R)^2}}$$

From Ohm's law:

$$V = IR \iff R = \frac{V}{I}$$

$$\text{Impedance}, Z = \frac{V}{I}$$
$$\text{Impedance}, Z = V \times \frac{\sqrt{(X_L - X_C)^2 + (R)^2}}{V}$$
$$\textbf{Impedance}, \mathbf{Z} = \sqrt{(X_L - X_C)^2 + (R)^2}$$

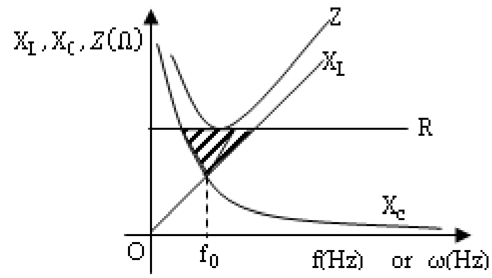
Phase angle: is the angle between the resultant voltage and the current flowing through the circuit.

$$\tan\Phi = \frac{V_L - V_C}{V_R} = \frac{I(X_L - X_C)}{IR} = \frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}$$

$$= \frac{\omega^2 LC - 1}{\omega RC}$$

$$\mathbf{tan}\,Ö = \frac{\omega^2 LC - 1}{\omega RC} \iff Ö = \mathbf{tan}^{-1}\left(\frac{\omega^2 LC - 1}{\omega RC}\right)$$

**A graph of R,$X_L$ and $X_C$ against frequency, f**




$f_0$ = Resonant frequency

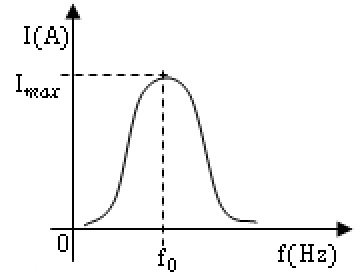
**Electrical resonance**

Electrical resonance is achieved when the voltage across the inductor is equal to the voltage across the capacitor for an R.C.L series circuit.
At this point, the total impedance Z= R. Therefore, the current I flowing is maximum and it oscillates at a resonant frequency, $f_0$.
A series resonant frequency is called an acceptor circuit.

At $f_0$, $V_L = V_C \iff IX_L = IX_C$
$\iff X_L = X_C$

$$\iff \omega L = \frac{1}{\omega C}$$
$$2\pi f_0 L = \frac{1}{2\pi f_0 C}$$
$$2\pi f_0 L(2\pi f_0 C) = 1$$
$$4\pi^2 f_0^2 LC = 1$$
$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

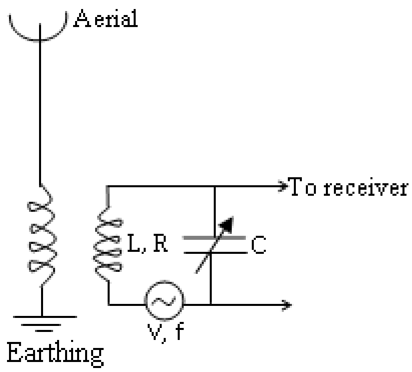
A graph of I against f for an R-C-L series circuit




From the graph, Current, I is maximum at the resonant frequency,$f_0$. Thus maximum current flows in an RCL series circuit when the circuit is resonating. $X_L = X_C$ and Z has its maximum value equal to R.

Application of electrical resonance

R-C-L series circuits are applied in tuning of a radio receiver. The radio station is heard clearly and loudest at the resonant frequency.




-The incoming wave signal of frequency, $f_0$ from a distant transmitting station induces a varying voltage in the aerial which in turn induces a voltage, V in the coil of inductance, L and the same frequency, $f_0$.

-The capacitance, C is varied by tuning hence varying the resonant frequency, f and at one setting, the resonant frequency $f = f_0$ the frequency of the incoming wave. Maximum current is then obtained and the station is heard loud and clear.

**Examples:**

| Radio station | Resonant frequency(Hz) |
|---|---|
| 1. K.F.M | 93.3 |
| 2. Capital F.M | 91.3 |
| 3. Simba F.M | 97.3 |

**Example 1:**

A circuit consists of a 200μH inductor of resistance10Ω in series with a variable capacitor and a 0.10V [r.m.s], 1.0MHz supply. Calculate the;

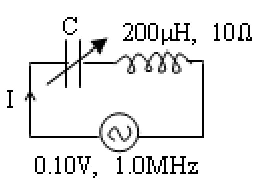
(i) Capacitance to give resonance

(ii) P.d across the inductor

(iii) P.d across the capacitor.

**Solution:**

**(i)**




$\mathrm{L} = 200\mu\mathrm{H} = 200 \times 10^{-6}\mathrm{H}, \mathrm{R} = 10\Omega$

$\mathrm{f_0} = 1.0\mathrm{MHz} = 1.0 \times 10^{6}\mathrm{Hz}.$

At resonance; $V_L = V_C$

But also; $f_0 = \dfrac{1}{2\pi\sqrt{LC}}$

$$C = \frac{1}{4\pi^2 f_0^2 L}$$

$$C = \frac{1}{4\pi^2(1.0 \times 10^6)^2(200 \times 10^{-6})}$$

$$C = 0.00013\mu F$$

(ii)

From; $V = IZ$

At resonance; $V_L = V_C$ and the impedance is equal to resistance. i.e, $Z = R = 10\Omega$.

If V is the applied p.d (0.1V), then the current I is given by;

$$V = IR \Leftrightarrow I = \frac{V}{R} = \frac{0.1}{10} = 1.0 \times 10^{-2}\mathrm{A}$$

If the inductor has reactance, $X_L$, the p.d $V_L$ across it is given by;

$V_L = IX_L = I(ù_0 L) = I(2ð f_0 L)$

$V_L = (1.0 \times 10^{-2})(2ð \times 1.0 \times 10^{6})(200 \times 10^{-6})$

$V_L = 4\pi\ V$

(iii)

At resonant frequency, $V_L = V_C$

Hence; $V_C = 4\pi\ V$

**Example 2:**

A coil with self inductance 0.8H and having a resistance of 25Ω is connected in series with a 4μF capacitor and a.c. mains of 240V, 50Hz. Find the;

(i) Circuit impedance.

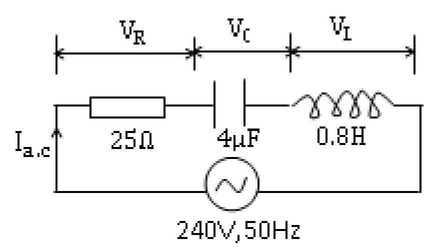
(ii) r.m.s value of the p.d across the capacitor and coil.

(iii) Power dissipated in the circuit.

(iv) Phase angle between the applied voltage and the current flowing.

**Solution:**

**(i)**



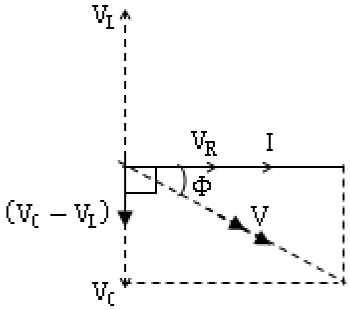

Inductive Reactance, $X_L = 2ð f L$

$X_L = 2ð f L = 2ð \times 50 \times 0.8 = 251.33\Omega$

Capacitive Reactance, $X_C = \frac{1}{2\pi fC}$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 4 \times 10^{-6}} = 795.77\Omega$$

Thus, since $X_C > X_L$, then $V_C > V_L$ and hence the circuit is purely capacitive, and the resultant voltage is in the direction of $V_C$.




$V^2 = (V_C - V_L)^2 + {V_R}^2$
$V^2 = (IX_C - IX_L)^2 + (IR)^2$
$V^2 = I^2[(X_C - X_L)^2 + (R)^2]$

$$I = \sqrt{\frac{V^2}{[(X_C - X_L)^2 + (R)^2]}}$$

$$I = \frac{V}{\sqrt{(X_C - X_L)^2 + (R)^2}}$$

$\text{Impedance}, Z = \sqrt{(X_C - X_L)^2 + (R)^2}$
$Z = \sqrt{(795.77 - 251.33)^2 + (25)^2}$
$Z = 545.01\Omega$

(ii)
From Ohm's law;

$$V_{r.m.s} = I_{r.m.s}Z \Leftrightarrow I_{r.m.s} = \frac{V_{r.ms}}{Z}$$

$$I_{r.m.s} = \frac{240}{545.01} = 0.44\text{A}$$

R.m.s p.d across the coil
$V_{r.m.s} = I_{r.m.s}X_L$
$V_{r.m.s} = (0.44)(251.33)$
$V_{r.m.s} = 110.6\text{V}$

R.m.s p.d across the 25Ω in the coil
$V_{r.m.s} = I_{r.m.s}R$
$V_{r.m.s} = (0.44)(25)$
$V_{r.m.s} = 11\text{V}$

Thus resultant p.d across the coil is given by;
$V = \sqrt{(110.6)^2 + (11)^2} = 111.1\text{ V}$

<u>Alternatively:</u>
The impedance of the coil is given by;
$Z_{coil} = \sqrt{(X_L)^2 + (R)^2}$
$Z_{coil} = \sqrt{(251.33)^2 + (25)^2}$
$Z = 252.57\Omega$
Then;
$V_{r.m.s} = I_{r.m.s}Z_{coil}$
$V_{r.m.s} = (0.44)(252.57)$
$V_{r.m.s} = 111.1\text{V}$

R.m.s p.d across the capacitor
$V_{r.m.s} = I_{r.m.s}X_C$
$V_{r.m.s} = (0.44)(795.77)$
$V_{r.m.s} = 350.14\text{V}$

(iii) Power is dissipated in the resistive part of the circuit only. Thus;
$\text{Power}, P = I^2_{r.m.s}R$
$\text{Power}, P = (0.44)^2 \times 25$
$\text{Power}, P = 4.84\text{ W}$

(iv) Phase angle

$$\tan\Phi = \frac{V_C - V_L}{V_R} = \frac{X_C - X_L}{R}$$

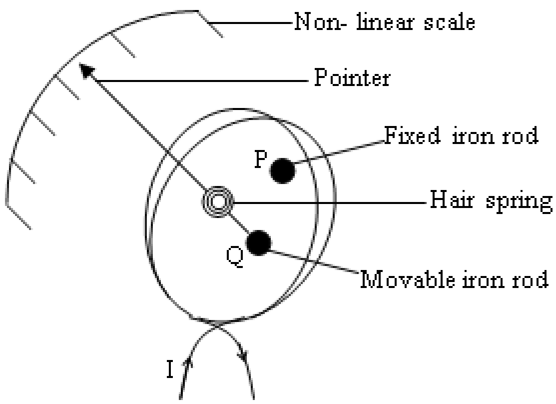
$$\Phi = \tan^{-1}\left(\frac{X_C - X_L}{R}\right)$$
$$= \tan^{-1}\left(\frac{795.77 - 251.33}{25}\right)$$
$$\Phi = 87.4^0$$

**Instruments used to measure a.c**
Moving coil meters cannot be used to measure alternating current. This is because;
In moving coil meters, action depends on the torque exerted on the coil of the wire, which also depends on the magnitude and direction of current.
When an a.c is passed through the coil, a varying torque will be exerted on the coil which causes the pointer to vibrate or oscillate with small amplitude about amean position at the frequency of the supply a.c. mains.
Hence a steady reading cannot be obtained.

Instruments used to measure a.c depend on the effect of a.c through a conductor, such as; the <u>Magnetising effect</u> and the <u>Heating effect.</u>

(a) Magnetising effect dependent instruments
Moving Iron Meters
(i). Repulsion type

<u>Moving Iron Ammeter (Repulsive type)</u>
<u>Structure:</u>
The repulsive type of a moving iron ammeter consists of two rods of soft iron, one fixed rigidly in position and the other attached to a pivoted pointer inside a solenoid carrying current to be measured.




Action:

-The current I to be measured is passed through solenoid. Whatever the direction of the current, the iron rods are magnetized in the same sense and hence repel each other.

-The repulsive forces cause the movable iron rod Q and hence the pointer to move and deflect along the non linear scale until it is stopped by the restoring couple due to the hair springs.

-For good approximation, the magnetic force of repulsion between the iron rods is directly proportional to the mean value of the square of current. Hence the deflection of the pointer is proportional to the mean value of the square of current.

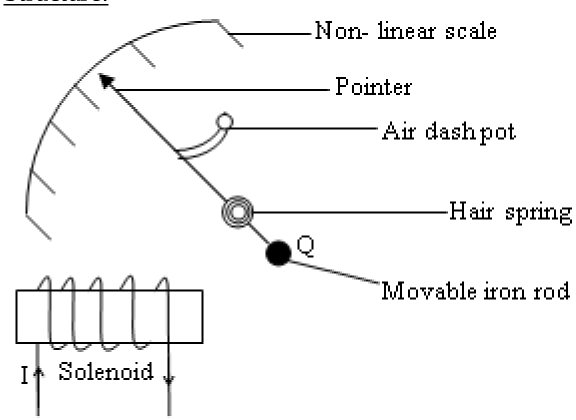
$$\grave{e} \propto \langle I^2 \rangle = I_{r.m.s}^2$$

-The scale reads root mean square values of current I and hence its a non-linear scale.

(ii). Attraction type.

Moving Iron Ammeter (Attraction type)

Structure:




The attraction type of a moving iron ammeter consists of a solenoid, a movable iron rod attached to the pointer with an air dashpot, and hair springs which control the rotation of the pointer along a non linear scale.

Action:

-The current I to be measured is passed through solenoid hence magnetizing it. Whatever the direction of the current, the solenoid is magnetized and attracts the movable soft iron attached to the pointer.

-The attraction forces cause the movable iron rod Q and hence the pointer to move and deflect along the non linear scale until it is stopped by the restoring couple due to the hair springs.

-The air dashpot traps air in the pointer and prevents over damping of the pointer.

-For good approximation, the magnetic force of attraction between the solenoid and the movable iron rod, Q is directly proportional to the mean value of the square of current.

Hence the deflection of the pointer is proportional to the mean value of the square of current.

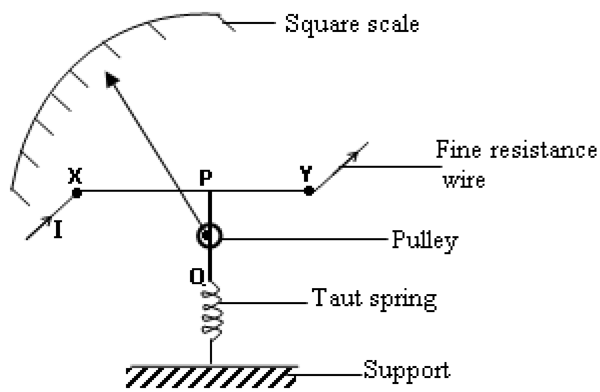
$$\grave{e} \propto \langle I^2 \rangle = I_{r.m.s}^2$$

-The scale reads root mean square values of current I and hence its a non-linear scale.

(b) Heating effect dependent instruments

(i). Hot wire meter (Hot wire ammeter)

Structure:




Action:

-Current to be measured is passed through the resistance wire XY which heats up. The rise in temperature of the wire makes it expand and sag.

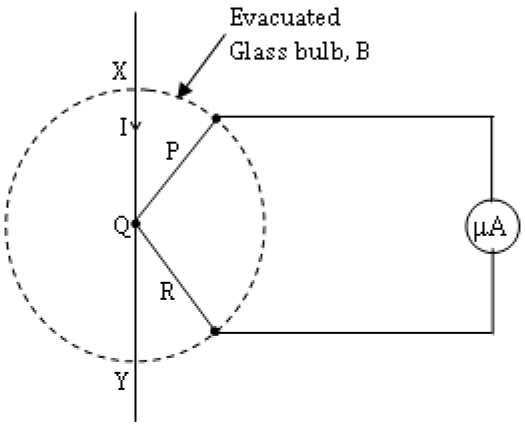
-The sag is transmitted to a second fine wire PQ, which is held taut by a spring. The wire PQ passes round a pulley attached to the pointer of the instrument, which rotates as the wire XY sags.

-The deflection of the pointer is roughly directly proportional to the average rate at which heat is developed in the wire XY; it is therefore proportional to the average value of the square of the alternating current.

$$\theta \propto \langle I^2 \rangle = I_{r.m.s}^2$$

(ii). Thermo couple meter

Structure:




It consists of two wires of dissimilar metals P and R joined together at Q (the centre of XY) which form one junction of the thermocouple.
The other junction is at room temperature.
The junction Q is enclosed in an evacuated bulb, B to shield it from draught.

<u>Action:</u>
The current to be measured is passed through wire XY which heats up and thus raises the temperature of junction Q.
-A thermo electric e.m.f is thus generated which causes a direct current to flow. This current is measured by a moving coil micro meter previously calibrated to read root mean square values of current.
**Note**: A thermocouple meter is used to measure very high frequency a.c. This is because it has very low inductance and capacitance.

**Differences between Moving coil meter and moving iron meter**

| **Moving coil meter** | **Moving Iron meter** |
|---|---|
| -Action depends on torque exerted on current carrying rectangular coil in magnetic field | -Action depends on the magnetising effect of current through a conductor. |
| -Measures mean values, hence can only be used to measure d.c. | -Measures root mean square values, hence can be used to measure both d.c and a.c. |
| -Has a linear scale. | -Has a non linear scale. |

**Advantages of moving iron ammeter**
- they are cheap
- It can be used to measure both a.c and d.c.

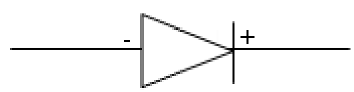
**Disadvantages of moving iron ammeter**
- They have non-linear scale.

**Similarities between a.c and d.c.**
Both can be cause:
-Magnetisation
-Heating
-Lighting

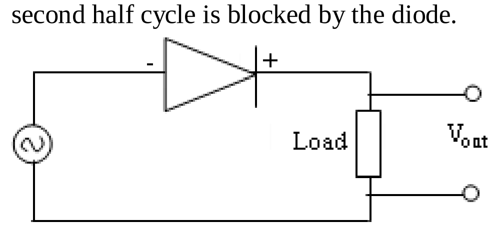
**Differences between a.c and d.c.**

| **D.c** | **A.c** |
|---|---|
| -Can be used in electrochemical processes. E.g electro plating | -Ac is useless in this aspect |
| -Can be used in electric trains for locomotion. | -The train would simply move forward and backwards at the frequency of the a.c supply. |
| -Can't be stepped up or down | -Can easily be stepped up and down by using transformers. |
| -cannot | -Can be transported for long distances with minimum power loss |
| -D.c can't be conducted by capacitors | -A.c can be conducted by capacitors |
| -D.c is already rectified | -A.c can easily be converted to d.c using rectifiers |

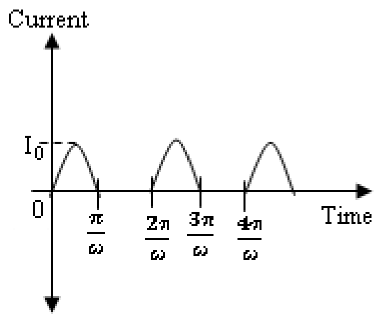
**RECTIFICATION**
Rectification is the process by which a.c is converted to d.c.
During rectification, a diode which shows low resistance to the flow of current in one direction and a very high resistance to current flow in the opposite direction is used.




**Types of Rectification:**
**(a) Half-wave rectification**
This is where a.c is converted to d. c such that current in the second half cycle is blocked by the diode.




-When current flows clockwise, the resistance of the diode is low.
-In the 2$^{nd}$ half cycle, when current would be flowing in the opposite direction (anti- clockwise), the resistance of the diode is very high and so current is switched off.
-The energy in the switched off half cycle appears as heat energy and warms up the diode.



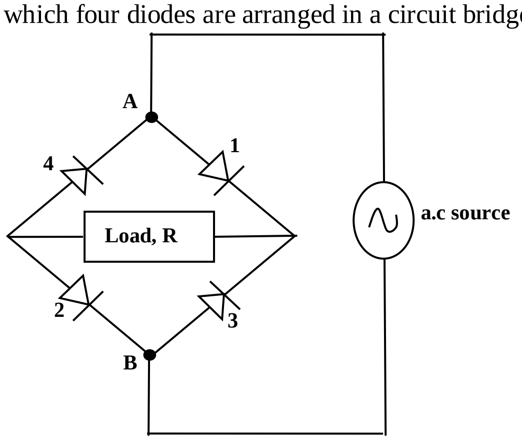

A moving coil galvanometer can be used to measure the average value of the current, $< I >$ .
From the definition of mean value;

$$< f(t) >= \frac{1}{T}\int_0^T f(t)dt$$

$$< I >= \frac{\omega}{2\pi}\int_0^{\frac{\pi}{\omega}}(I_0 \sin \omega t)dt$$

$$< I >= \frac{I_0}{\pi} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (i)$$

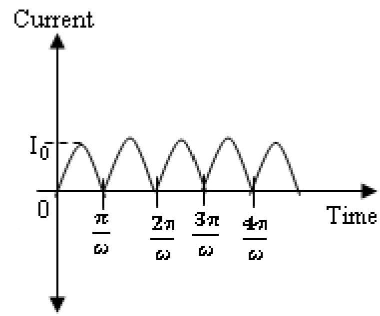
**(b) Full-wave rectification:**
Although current has been rectified and made to flow in one direction, during half wave rectification, half of the energy is lost.
To over come this problem, we use full- wave rectification in which four diodes are arranged in a circuit bridge below.




- During the 1st half cycle, point A is positive relative B. Thus current flows through diodes 1 and 2. Diode 2 takes back the current to the source.

- During the 2nd half cycle, point B is made positive relative B. Thus current flows through diodes 3 and 4. Diode 4 takes back the current to the source.

- Thus there is always a current flowing in the same direction through the load, R.




A moving coil galvanometer can be used to measure the average value of the current, $< I >$ .
From the definition of mean value;

$$< f(t) >= \frac{\omega}{\pi}\int_0^T f(t)dt$$

$$< I >= \frac{\omega}{\pi}\int_0^{\frac{\pi}{\omega}}(I_0 \sin \omega t)dt$$

$$< I >= \frac{2I_0}{\pi} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (ii)$$

This is twice that in half wave rectification.

# ELECTROSTATICS

This deals with the study of charges at rest.

**Types of charge**

An amber rod rubbed with dry wool for attracts small pieces of paper. A glass rod rubbed with dry silk attracts small bits of paper. The rods are said to be electrified (to acquire charge) through the rubbing process.

Two amber rods rubbed with fur repel each other; similarly two glass rods rubbed with silk repel each other. An amber rod rubbed with fur attracts a glass rubbed with silk.

The above observations show that:-

(i). Unlike charges attract whereas like charges repel
(ii). There exist two types of charges

The charge that appears on a glass rod rubbed with silk has been labeled positive and that which appears on an amber rod rubbed with fur is negative.

**Conservation charge**

The algebraic sum of the electric charges in any closed system remains constant. Uncharged objects contain equal amounts of negative and positive charge. When a glass rod is rubbed with silk, negative charge is transferred from the glass rod to the silk, leaving the glass rod with equal and opposite charge. For the closed system consisting of a glass rod and silk, the algebraic sum of the electric charge is constant.

**Electrons and atoms**

An atom consists of a positively charged, sense nucleus, consisting of protons and neutron. Reading about the nucleus are the electrons. The charge of the electron is negative and equal to $-1.6\times10^{-19}C$. It is the smallest charge, it is possible to obtain. The charge on the proton is $1.6\times10^{-19}C$.

The number of proton in the nucleus of an atom is called the atomic number and it is denoted by Z. Hence a neutral atom has positive charge equal to $^{+}Ze$ and positive charge equal to $^{-}Ze$ where $e = 1.6\times10^{-19}C$.

The mass of an electron is $9.11\times10^{-3}kg$ and this is about $\frac{1}{1840}$ of the mass of the proton. The neutron has no charge; its mass is slightly greater than the mass of a proton.

**Conductors, insulation and semi conductors**

Atoms in solids are closely packed. The nuclei are separated by distances of $10^{-10}$m. In conductors, most of the electrons are bound to the parent nuclei but about one electron per atom is free to wander through the lattice and can easily drift in one atom under an extreme electric field.

Insulators or di-electrics are materials in which all atomic electrons are bound to their potent nuclei. Electrons can be removed or added to an insulator only by expenditure of larger amounts of energy.

Semi-conductors lie between conductors and insulators in their conduction properties. In semi-conductors, a few electrons are free; the no of free electrons can be increased by heating.

| conductors | semi-conductors | insulators |
|---|---|---|
| Metals | Silicon, Germanium | Vacuum, plastics |

**Charging by rubbing or by friction**

When two unlike di-electric are rubbed together, heat is produced. The thermal energy is sufficiently to cause removal of the weakly bound electrons in one material ( i.e. One with a lower work function). The former is left with net positive charge while the latter (one with a higher work function) receives the electrons and acquires net negative charge of equal magnitude as the positive charge.

The two substances are thus said to have acquired charge by rubbing as a result of electron transfer.

**Charge conservation**

The total electric charge on any object is on integral multiple of electrons since the charge is simply the algebraic sum of the charges of the elementary particles that make up the body. The charge on an elementary particle is either positive, zero or negative. Hence the charges scale equipment's because the experiments involve a large number of electrons.

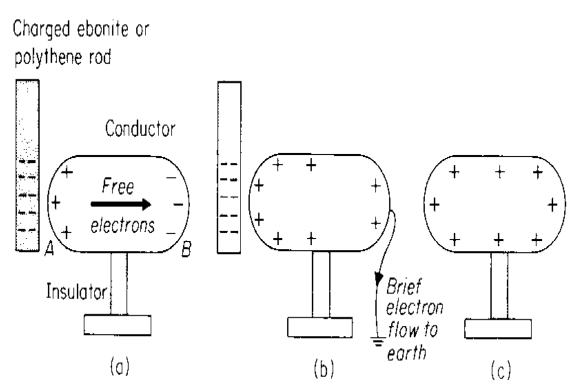
**Electrostatic induction**

This is the process by which an un charged body acquires charge from a charged body placed in its vicinity and not in contact.

**Charging by electrostatic induction**

Electrification by induction

(a) Charging the body positively.

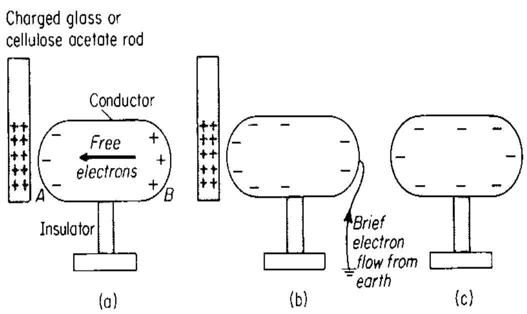



Procedure

- Put the conductor on an insulated stand as in (a)
- Bring a negatively charged rod near the conductor. The positive and negative charges separate as shown in (a)

- Earth the conductor by momentarily touching it with a finger and electrons flow from it to the earth as in (a) in presence of the charging rod.
- Remove the charged rod, When the rod is withdrawn, the positive charge on the sphere distributes itself over the entire surface of the conductor.
- The conductor is found to be positively charged.

(b) Charging the body by induction negatively,

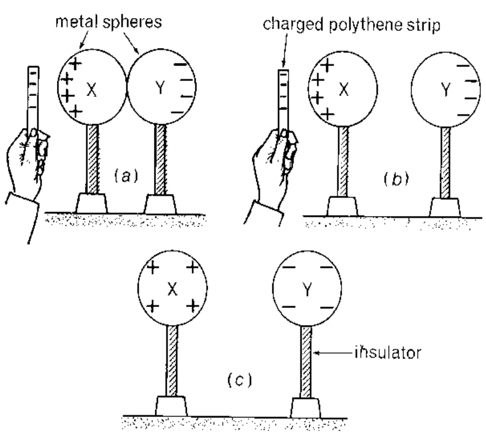



Procedure

- Put the conductor on an insulated stand as in (a)
- Bring a positively charged rod near the conductor. The positive and negative charges separate as shown in (a)
- Earth the conductor by momentarily touching it with a finger and electrons flow from it to the earth as in (b) in presence of the charging rod.
- Remove the charged rod, When the rod is withdrawn, the negative charges on the conductor distribute themselves over the entire surface of the conductor.
- The conductor is found to be negatively charged.

**Charging two conductors simultaneously:**

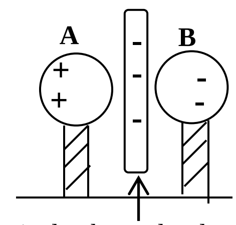
**(i). So that they acquire opposite charges.**




**Explanation**

- Support two uncharged bodies on an insulated stand as shown in (a)
- Bring negatively charged amber or ebonite rod near the two bodies, positive and negative charges separate as in (b). Negative charge on the rod repels the loosely bound electron to the far side of Y.
- Separate (X) from (Y) in presence of the inducing charge (charging rod).
- Remove the inducing charge, (charging rod). (X) will be positively charged and (Y) will be negatively charged
- Tests show that A carries positive charge while B carried an equal amount of negative charge.

**(ii). So that they acquire opposite charges**




Negatively charged rod

- Two identical metal spheres A and B are supported by insulating stands.
- A negatively charged rod is placed between the two metal spheres.
- Positive charges in each sphere are attracted towards the negatively charged rod and negative charges (electrons) are repelled to the side remote to the charging rod.
- In presence of the charging rod, both conductors are earthed at the same time by touching the sides remote to the charging rod.
- On earthling the sphere, electrons flow to the ground. When the earth is disconnected, the radial spheres are left with positive charge.
- When the charging rod is withdrawn, the positive charge on the spheres distributes themselves over the entire surface of the sphere.

**THE GOLD LEAF ELECTROSCOPE.**

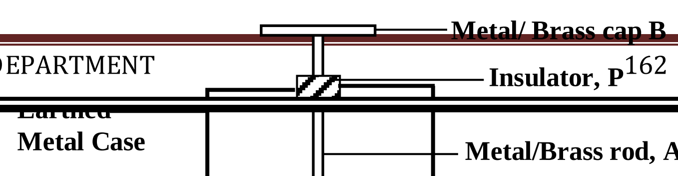
A gold leaf electroscope is an instrument used to detect presence of charge, test the sign of charge, compare the magnitude of charge on various bodies and test the insulating properties of a substance.

**Structure**

It consists of a metal rod A, to which a gold leaf or a very thin aluminum foil L is attached. The metal rod is fitted with a circular metal cup or disc B and is insulated from the metal casing C by means of a plug P.

The metal case is earthed in order to screen the electroscope from outside influences other than those brought near the cap.

The casing has glass or pespex windows through which the gold leaf may be observed.



**Mode of action**

When a charged body is brought near or in contact with the cap of the electroscope, the cap will acquire an opposite charge to that on the body by induction.

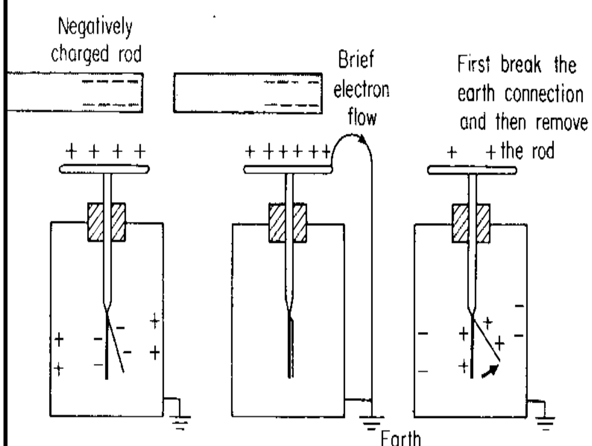
The charge on the body will repel all charges similar to it down to the metal rod, to the plate and the leaf.

Due to presence of like charges on the plate and gold leaf, the leaf diverges as it is repelled by the plate.

Leaf divergence implies that the body brought near or in contact with the cap carries a charge

**Charging a gold leaf electroscope.**

(i) Charging it positively




- Bring a negatively charged rod near the cap of a neutral gold leaf electroscope.
- Positive charges are attracted to the cap and negative charges are repelled to the plate and gold leaf.
- The leaf diverges due to repulsion of the same sign of charges on the plate and leaf.
- Earth the gold leaf electroscope in presence of a negatively charged rod.
- Electrons on the plate and leaf flow to the earth.
- The leaf collapses.
- Remove the negatively charged rod, positive charges on the rod spread out to the rod and leaf therefore the leaf diverges hence the gold leaf is positively charged.

(ii) Charging it negatively.

- Get an uncharged gold leaf of electroscope.
- Bring the positively charged rod near the gold leaf cap.
- Negative charges are attracted to the cap and positive charges are repelled to leaf and glass plate.
- Earth the gold leaf electroscope in presence of a positively charged rod.
- Negative charges flow from the earth to neutralize positive charges on plate and leaf.
- The leaf collapses.
- Remove the positively charged rod, negative charges on the cap spread out on the leaf plate, therefore, the leaf diverges and a gold leaf therefore becomes negatively charged.

**Uses of a Gold leaf Electroscope**

1. To detect the presence of charge on a body.

Bring the body under test near the cap of a neutral G.L.E.

If the leaf deflects, then the body has got a charge.

However, if the leaf remains un deflected, then the body is neutral (has no net charge).

2. To test the nature or sign of charge on a body.

Bring the body under test near the cap of a charged G.L.E.

If the leaf diverges further, then the body has a charge similar to that on the G.L.E.

However, if the leaf collapses, then the body is either neutral or it carries a charge opposite to that on the G.L.E. In this case, we can not conclude. But the G.L.E is discharged by touching its cap with a finger and then given a charge opposite to the one it had previously and the experiment is repeated.

If still the leaf collapses, then the body is neutral.

**NOTE: An increase in leaf divergence is the only sure test for the sign of charge on a body.**

**Increase in leaf divergence occurs when the test charge and the charge on the gold leaf electroscope are the same.**

3. To compare and measure potentials.

Two bodies which are similarly charged are brought in contact with the cap of a G.L.E one after the other.

The divergences in the two cases are noted and compared.

The body which causes more divergence is at a higher potential.

4. To classify conductors and insulators.

Bring the body under test in contact with the cap of a charged G.L.E.

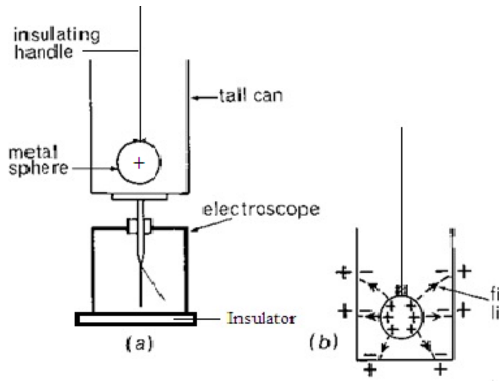
If the leaf collapses suddenly, then the body is a good conductor.

If the leaf collapses gradually, then the body is a poor conductor

If the leaf does not collapse, then it is an insulator.

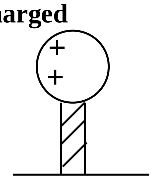
**Faraday's ice-pail experiment**

The experiment reveals the manner in which charge is distributed on hollow conduction.




- A metal sphere is suspended by an insulating thread is given positive charge.
- The sphere is then lowered into a metal can C connected to an uncharged electroscope as shown in (a). The leaves of the electroscope are observed to diverge.
- The metal sphere is shifted to various positions inside C but without touching C. the divergence θ on the electroscope is observed to remain the same.

**Conductor is positively charged**



- The sphere is withdrawn from C. the leaf of the electroscope collapses. The sphere is again lowered into C. the leaf of the electroscope is observed to diverge by the same amounts as before.
- The sphere is then allowed to touch the inner surface of the can. The divergence θ is observed to remain constant.
- The sphere is then withdrawn and tested for charge. It is found to have lost all its charge, showing that there must have been an equal negative charge on the inner surface of the can which neutralized the positive charge on the sphere.

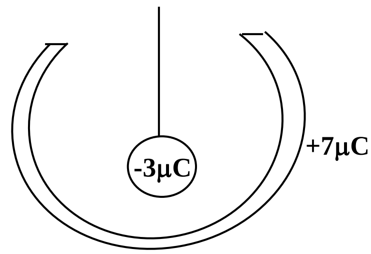
**Conclusions:**

For a hollow closed conductor;

(i) When a charged body is enclosed by a hollow conductor, it induces on the inside of the conductor equal but opposite charge.

(ii) The total charge inside a hollow conductor is always zero, either there are equal and opposite charges on the inside walls and than the volume (as was the case before the Sphere touched the can) or there is no charge at all. Any net charge on a hollow conductor resides on the outside surface of the conductor.

**Example**




A hollow spherical conductor carries a charge 7μC. A Pith having a charge of -3μC is introduced inside the spherical conductor in such a way that it does not touch the conductor. What happens when the pith ball is made to touch the inside of the conductor?

Positive charge +3μC is induced on the inside of the sphere. Negative charge of -3μC is induced on the outside of the sphere.

Hence the total charge on the outer surface is

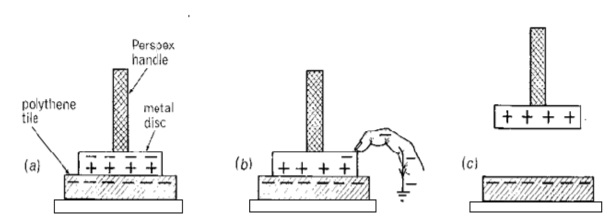
$[+7+(-3)]\mu\mathbf{C} = \mathbf{4}\mu\mathbf{C}.$

The positive charge on the inside surface of the sphere is neutralized, when the ball touched the surface. This leaves only the positive charge of 4μC on the outside of the conductor.

**The Electrophorus or Proof Plane**

This is a device which provides large quantities of charge by induction. It consists of an insulator (e.g a polythene tile) and a metal disc with an insulated handle.

It is a device used for converting mechanical energy to electrical energy. This is because work is done in raising the disc against the attraction of the opposite charges.




Procedures.

The polythene is charged negatively by rubbing it vigorously with dust.

When the metal disc is laid upon it, it acquires induced positive charge after earthling it with a finger.

Very little negative charges escape from the polythene to the disc. This is because the disc material has un even surfaces preventing it from touching at more than a few points. Little charge escapes from these points because polythene is a non conductor.

On removing the disc, it has sufficient positive charge.

The disc can be charged and charged again repeatedly until the charge on the polythene has disappeared by leakage.

Advantages of the electrophorus in charging.

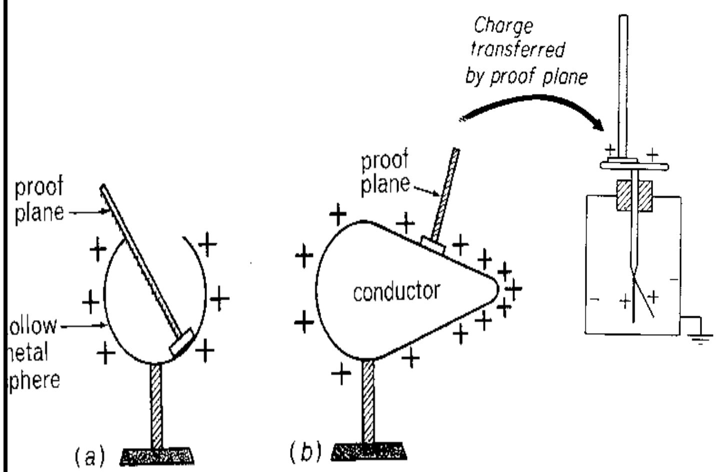
(i) Supply of charge is almost un limited

(ii) A greater charge almost equal to that of the whole polythene can be concentrated on the conducting disc.

(iii) Only very little charge can be transferred by contact leakage since the polythene is a non conductor.
(iv) The disc can be discharged and recharged.

**Distribution of charge on the conductor**

(a) Hollow conductor

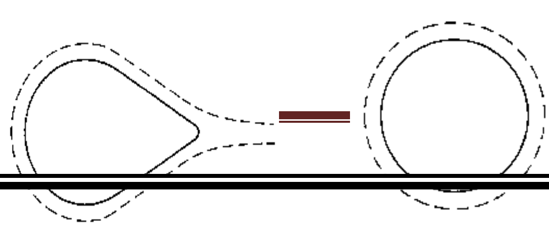



When the proof plane is placed on the outside surface of a charged hollow conductor, charge is transferred to the uncharged G.L.E, the leaf diverges as shown in (a). This proves that charge was present on the outside of the surface.
When the proof plane is placed on the inside of a charged conductor is transferred to the uncharged G.L.E, the leaf does not diverge as in (b) therefore, charge resides on the outside surface of the hollow charged conductor.

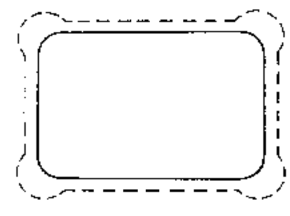
(b) Curved bodies

When the proof plane is placed on the surface of the conductor, a sample charge is acquired by the proof plane.
The plane is then transferred into a hollow can on electroscope and the deflection noted. The proof plane and electroscope are discharged.
Samples of charge are picked from different parts of the conductor and in each case the deflection on the electroscope noted.
In this way it is discovered that the surface density of charge on an unsymmetrical conductor is greatest where the curvature is greatest, that is, where the radius of curvature is least.

A curve with a big curvature has a small radius and a curve with small curvature has big radius therefore, curvature is inversely proportional to radius. A straight line has no curvature.
Surface charged density is directly proportional to the curvature. Therefore a small curvature has small charge density. Surface charge density is the ratio of charge to the surface area.



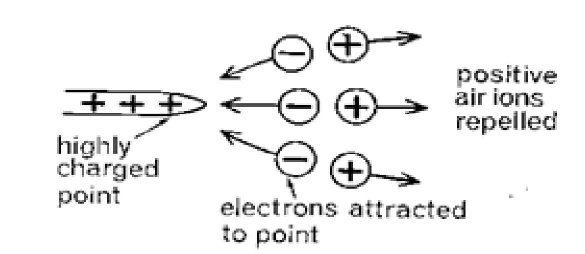
(ii)Spherical conductor




In the figures, the distance of the dashed curve from the surface of the conductor is proportional to the density of charge on the surface.
There is high density of charge at the pointed end of the conductor.

**Action of points and Corona discharge**

Charge concentrates at sharp points. Because of high density of charge at pointed end, the electric field intensity there is very high. This ionizes the surrounding air molecules producing positive and negative ions. Ions which are of the same charge as that on the sharp points are repelled away forming an electric wind which may blow a candle flame and ions of opposite charge are collected or attracted to the sharp points. Some of the charge on the conductor is neutralized. Thus the conductor loses some charge.
This apparent loss of charge is called Corona discharge.




Therefore, a charged sharp point acts as;
(i) Spray off' of its own charge in form of electric wind.
(ii) Collector of unlike charges. The spray off and collecting of charges by the points is known as **corona discharge** (action of points.)

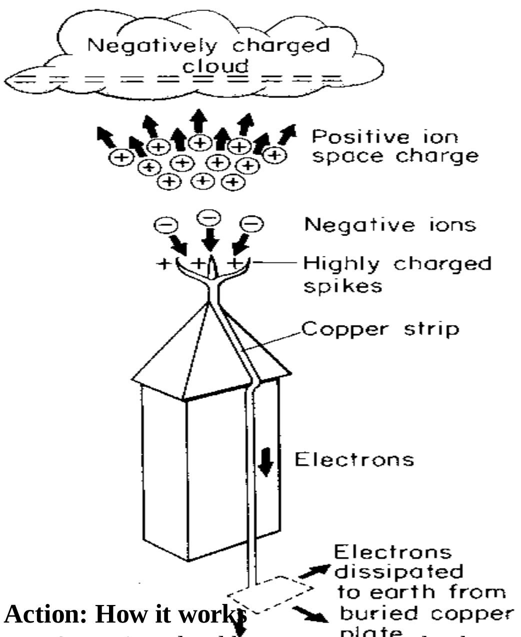
**Corona Discharge** is the apparent loss of charge by sharp conductors due the very high charge intensities at the pointed ends which causes ionization of air molecules around the pointed ends.

**Applications of corona discharge**

**(i) Electrostatic precipitator**
**(ii) Lightening conductor**

Lightening is a rapid, high – current discharge between clouds or between a cloud and the ground. In the latter case, the large current when if passes through a building it can cause the building to burn down. Trees split up under the expansion of steam produced in the tree when the tree is struck by lightening.
A lightening conductor is a sharp spiked conductor which is connected to a strip attached to a building and earthed.

A lightening conductor is made up of a thick copper strip which is fixed to the ground and on the walls of the tall building ending with several shaped spikes. It is used to protect structures from damage once struck by lightening.



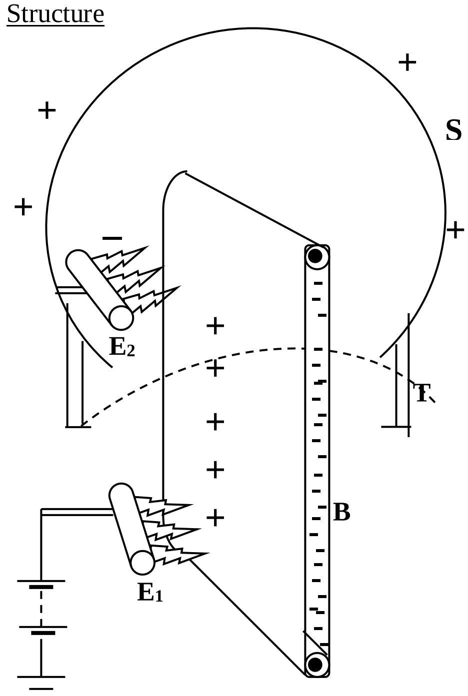

**Action: How it works**

- A moving cloud becomes negatively charged by friction.
- Once it approaches the lightening conductor, it induces opposite charge on the conductor.
- A high charge density at pointed ends creates a large electric field intensity which ionizes the air molecules and sends a stream of positively charged ions which neutralize some of the negative charges of the cloud.
- The excess negatively charged ions are safely conducted to the earth through a copper strip hence reducing the harmful effect of the cloud.

**How lightening causes damage to buildings and trees.**
In discharging the earth, lightening tends to strike the highest points such as the tallest tree among the trees or buildings.
Thunder clouds contain large quantities of negative charges on their underside and positive charges on their tops.
When a negatively charged cloud gathers over a building, it induces a large positive charge on the roof resulting in a very high potential difference the cloud and the building.
The force of attraction between the positive and negative charges can be so strong that the electrons suddenly jump off from the clouds to the roof producing a flash of lightening.
In the flash, without a lightening conductor, this current passes through a much higher resistance producing a great deal of heat.
The heat vapourises the moisture in the structure explosively and causes structural damage or some times sets the building on fire.

**(iii) The Van der Graff Generator**

Structure




**S - Metal hollow sphere**
**T -Insulting tube**
**B - Silk belt driven by a motor**
**$E_1$, $E_2$ - Metal electrodes of sharply pointed ends.**

It consists of a large hollow metal sphere S supported on an insulated tube T, A silk belt inside the tube is driven by an electric motor past a sharply pointed metal electrode, $E_1$ held at an electric potential of about $10^4$V relative to earth. As the belt moves upwards, it passes another sharply pointed metal electrode, $E_2$ connected on the inside of the hollow sphere.

Mode of operation

There is high electric field intensity at the sharp ends $E_1$. This ionizes the air there. The positive ions are repelled onto the belt.
The positive charge is carried up by the belt towards the sphere it induces negative charge on the sharp ends $E_2$ and positive charge on the sphere to which the blunt end of $E_2$ is connected.
The high electric field held at the pointed ends of $E_2$ ionizes the air there. A negative charge is repelled onto positive charge carried by the belt and neutralizes it before the belt passes over the upper pulley.
This process of the belt charging up and discharging is repeated many times per second and each time the belt passes,

the sphere S charges up positively until the electric potential is about $10^6$V relative to earth. The electrical energy acquired comes from the work done by the motor to move the belt against repulsion between positive charge on the sphere and positive charge on the belt.

**Causes of inefficiency of the Van der graaf generator.**

(i) Charge leakage: This is prevented by placing the metal hollow sphere on an insulating stand and enclosing the whole apparatus.

(ii) Efficiency of the motor: The higher the motor efficiency, the higher the efficiency of the Van der graaf generator.

**Guiding questions**

1. Describe an experiment to show that a charge resides only on the outside surface of the hollow conductor.

2. Describe the mechanism of charging by rubbing.

3. Describe what is meant by electrostatic screening.

4. Describe how a body can be charged but remains at zero potential.

## COULOMBS LAW OF ELECTROSTATICS

The force between two charged bodies is directly proportional to the product of the magnitudes of their charges and inversely proportional to the square of the mean distance between them.

$F \propto Q_1Q_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$

$F \propto \frac{1}{r^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$

Combining the two expressions gives;

$F \propto \frac{Q_1Q_2}{r^2}$

$F = k\frac{Q_1Q_2}{r^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$

Where, r is the mean distance between the two charged bodies

K is the proportionality constant.

$\boldsymbol{k} = \frac{\mathbf{1}}{\mathbf{4}ðå}$

Where, å =Permitivity of the medium in which the charges are placed.

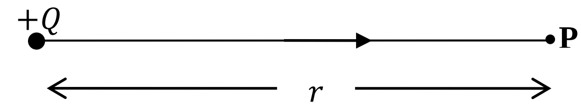
For free space or vacuum, $å = å_0$ (permittivity of free space)

$å_0 = \mathbf{8.85 \times 10^{-12} Fm^{-1}}$

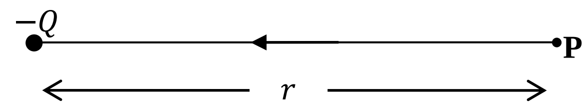
$\mathbf{k} = \frac{\mathbf{1}}{\mathbf{4}ðå_0} = \frac{1}{4\pi \times 8.85 \times 10^{-12}} = \mathbf{9 \times 10^9 mF^{-1}}$

**Note:**

**1**. The direction of force at a given point due to a positive charge is away from the positive charge towards the point.




**2**. The direction of force at a given point due to a negative charge is towards the negative charge from the point.




**Example 1.**

Two identical point charges repel each other with a force of $1.0 \times 10^{-5}$N. When the charges are moved 5mm further a part, the repulsive force reduces to $2.5 \times 10^{-6}$N.

(i) How far apart were the charges originally?

(ii) What is the magnitude of the charge of each?

Solution

(i)

Let the original separation be x

$F = k\frac{Q_1Q_2}{r^2} = k\frac{Q^2}{x^2}$

$1 \times 10^{-5} = k\frac{Q^2}{x^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$

When separation is increased by 5mm;

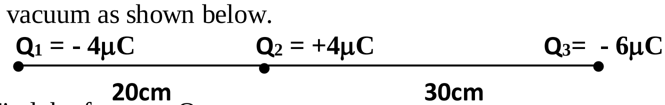
$2.5 \times 10^{-5} = k\frac{Q^2}{(x+0.005)^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$

Simplifying and solving equations (i) and (ii) you get

$\boldsymbol{x = 0.005}$ **m**

**Example 2.**

Three charges $Q_1$, $Q_2$ and $Q_3$ are arranged in a straight line in a vacuum as shown below.




Find the force on $Q_3$.

**Solution**

Let $F_1$ be the force induced on $Q_3$ due to $Q_1$

$F_2$ be the force induced on $Q_3$ due to $Q_2$

$$F_1 = \frac{kQ_1Q_2}{r_1^2} = \frac{9 \times 10^9 \times 4 \times 10^{-6} \times 6 \times 10^{-6}}{(0.5)^2} = 0.864\,\text{N}$$

(repulsive) towrds the left

$$F_2 = \frac{kQ_2Q_3}{r_2^2} = \frac{9 \times 10^9 \times 4 \times 10^{-6} \times 6 \times 10^{-6}}{(0.3)^2} = 2.4\,\text{N}$$

(attractive) towards the right .

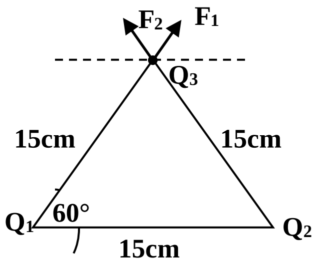
Resultant Force, or Net Force F on $Q_3$ is;

$F = F_2 + (-F_1)$

$F = 2.4 + (-0.864)$

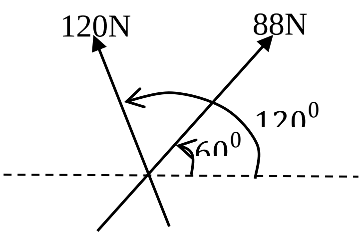
F = 1.536 N towards the right.

**Example 3.**
Three charges of $Q_1 = 11\mu C, Q_2 = 15\mu C$, and $Q_3 = 20\mu C$, are located at the corners of an equilateral triangle of side 15cm. Calculate the magnitude and direction of the net force on $Q_3$.




$$F_1 = \frac{kQ_1Q_3}{r_1^2} = \frac{9\times10^9\times11\times10^{-6}\times20\times10^{-6}}{(0.15)^2} = 88\,N$$

$$F_2 = \frac{kQ_2Q_3}{r_2^2} = \frac{9\times10^9\times15\times10^{-6}\times20\times10^{-6}}{(0.15)^2} = 120\,N$$




Considering the angle of each force measured from positive x-axis going anti-clockwise.

$$F_x = 88\cos 60 + 120\cos 120 = -16\,N$$

$$F_y = 88\sin 60 + 120\sin 120 = 180.133\,N$$

$$F = \sqrt{F_x^2 + F_y^x} = \sqrt{\left[(-16)^2 + (180.133)^2\right]} = 180.84\,N$$

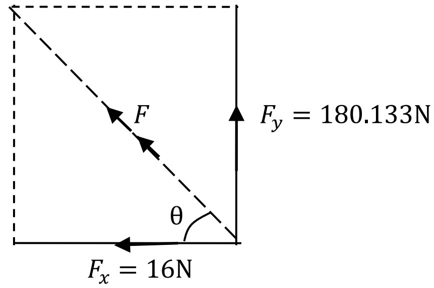
Alternatively, consider the acute angle and the sign of the trigonometric ratio in that quadrant where the force lies.

$$F_x = 88\cos 60 + -120\cos 60 = -16\,N$$

$$F_y = 88\sin 60 + 120\sin 60 = 104\sqrt{3}\,N$$

$$F = \sqrt{F_x^2 + F_y^x} = \sqrt{\left(16^2 + (104\sqrt{3})^2\right)} = 180.84\,N$$

Direction of the resultant force.



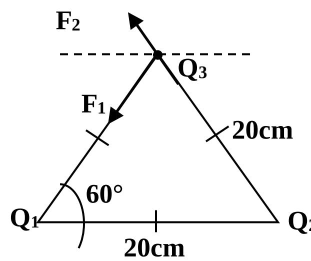

$$\tan\theta = \frac{F_y}{F_x}$$

$$\tan\theta = \frac{180.133}{16}$$

$$\theta = 84.9^0$$

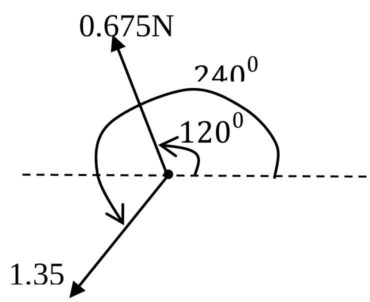
*Thus the resultant force is* 180.84*N at an angle of* $84.9^0$ *to the negative* $x - axis$.

**Example 4.**
Three charges of $Q_1 = -1\mu C, Q_2 = 2\mu C$, and $Q_3 = -3\mu C$, are located at the corners of an equilateral triangle of side 20cm. Calculate the magnitude and direction of the net force on $Q_3$.
**Solution**




$$F_1 = k\frac{Q_1Q_3}{r^2} = \frac{9\times10^9(1\times10^{-6})\times(3\times10^{-6})}{(20\times10^{-2})^2} = 0.675N$$

$$F_2 = k\frac{Q_2Q_3}{r^2} = \frac{9\times10^9(2\times10^{-6})\times(3\times10^{-6})}{(20\times10^{-2})^2} = 1.35N$$



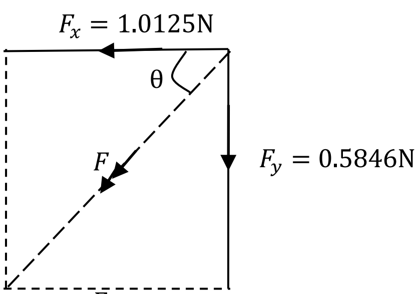

Considering the angle of each force measured from positive x-axis going anti-clockwise.

$$F_x = 0.675\cos 120 + 1.35\cos 240 = -1.0125\,\text{N}$$
$$F_y = 0.675\sin 120 + 1.35\sin 240 = -0.5846\,\text{N}$$
$$F = \sqrt{F_x^2 + F_y^x}$$
$$F = \sqrt{\left[(-1.0125)^2 + (-0.5846)^2\right]}$$
$$F = 1.1691\,\text{N}$$

Alternatively, consider the acute angle and the sign of the trigonometric ratio in that quadrant where the force lies.

$$F_x = -0.675\cos 60 + -1.35\cos 60 = -1.0125\,\text{N}$$
$$F_y = -0.675\sin 60 + 1.35\sin 60 = -0.5846\,\text{N}$$
$$F = \sqrt{F_x^2 + F_y^x}$$
$$F = \sqrt{\left((-1.0125)^2 + (-0.5846)^2\right)}$$
$$F = 1.1691\,\text{N}$$

Direction of the resultant force.

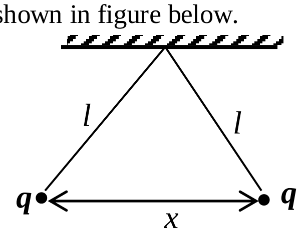



$$\tan\theta = \frac{F_y}{F_x}$$
$$\tan\theta = \frac{0.5846}{1.0125}$$
$$\theta = 30^0$$

*Thus the resultant force is* $1.1691N$ *at an angle of* $30^0$ *below the negative* $x-$*axis.*

**Questions**

1. Two identical conducting balls of mass m are each suspended in air from a silk thread of length, l when the two balls are each given identical charges q, they more apart as shown in figure below.




If at equilibrium each thread makes a small angle **θ** with the vertical, show that the separation ***X***, is given by:

$x = \left[\frac{q^2 l}{2\pi\varepsilon_0 g}\right]^{\frac{1}{3}}$; Where $\varepsilon_0$ is the permittivity of free space.

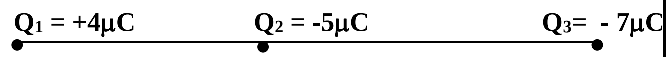
2. (a) Calculate the value of two equal charges, if they repel each other with a force of 0.1N when placed 50cm apart in a vacuum. [Ans: 1.667μC]

(b) What would be the size of the charges if they were situated in an insulating liquid whose permittivity is ten times that of a vacuum?[Ans: 5.27μC]

3. Find theforce between two point charges of +2μC and -3μC placed 10cm apart in air.

[Ans: 10.8N (Attraction)]

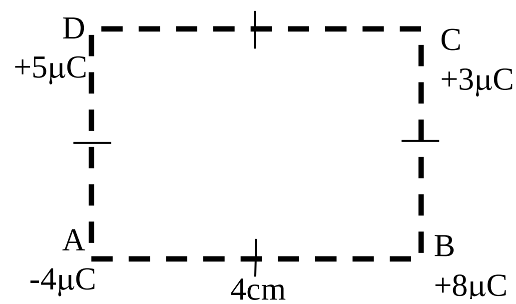
5.




Find the resultant force on $Q_2$.

[Ans: R=6.25N, towards $Q_3$]

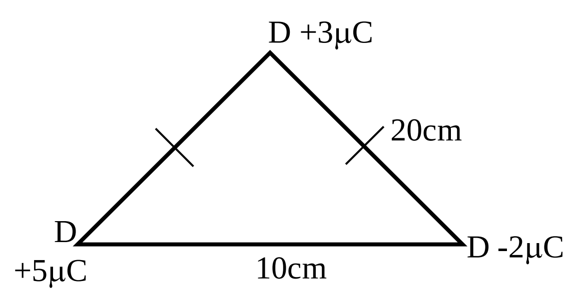
6. Four charges are placed at the corners of a square ABCD as shown below.



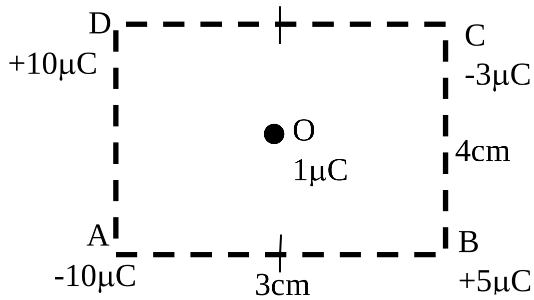

Find the resultant force on the charge at point C.

[Ans: R=126.54N at 61.4$^0$ to the horizontal.]

7. Find the resultant force at A




8. Find the resultant force at O.




[Ans: R=139.32N at 82.9$^0$ to the horizontal.]

9. Find the resultant force on $Q_3$.

[Ans: R=22.66N at 52.1$^0$ to the horizontal.]

## ELECTRIC FIELDS

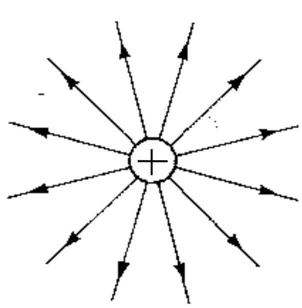
An electric field is a region where an electric force is experienced.
The direction of the electric field lines indicates the direction of the field. A higher density of the field lines indicate a strong electric filed.
A collection of lines of force is called Electric flux.

**Properties of electric field lines**

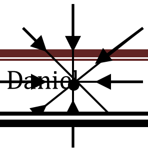
1. Each line of force start from the positively charged body and end at the negatively charged body
2. No two lines of force can cross each other
3. The density of lines of force at a point gives the direction of the electric field intensity at that point
4. Electric field lines are always normal to the surface of the conductor both when starting and ending on a conductor.

**Electric field pattern**

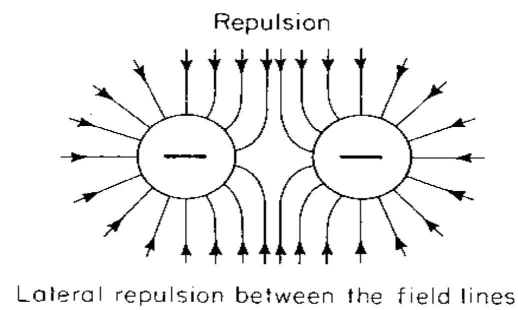
i. Due to an isolated point positive charge



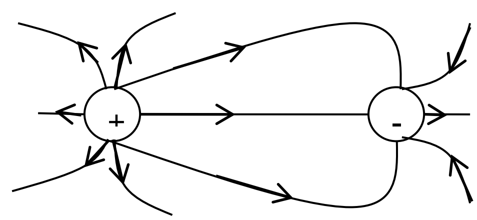
ii. Dues to an isolated negative charge



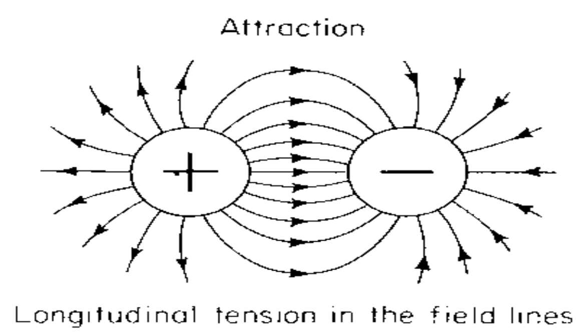
iii. Two equal charges of same sign



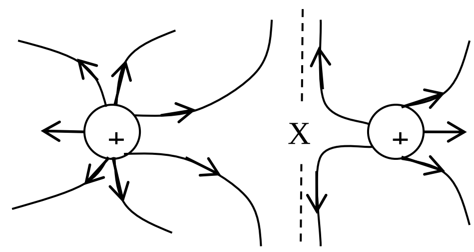

iv. Two unequal charges of opposite signs



v. Two equal charges of opposite signs




iv. Two unequal charges of the same signs




vi. A point positive charge near a parallel plate

**ELECTRIC FIELD INTENSITY/STRENGTH**

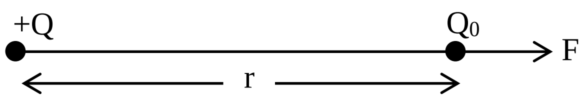
Electric field intensity of a point is the electric force exerted on one coulomb (1C) of a positive charge placed at a point in an electric field. i.e. force per unit positive charge.

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{\mathbf{Q}}$$

The direction of the electric field intensity is always away from the positive charge and towards a negative charge.
***Electric field intensity is a vector quantity.***

***Electric field intensity due to a charge***

Consider a charge $Q_0$ placed at a distance, ***r*** from a point positive charge, ***Q***




The force ***F*** *o*n $\boldsymbol{Q_0}$ is F;

$$F = k\frac{QQ_0}{r^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(i)$$

But Electric field intensity is Force per unit charge, Hence;

$$\vec{E} = \frac{F}{Q_0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(ii)$$

Substituting for F into equation (ii) gives;

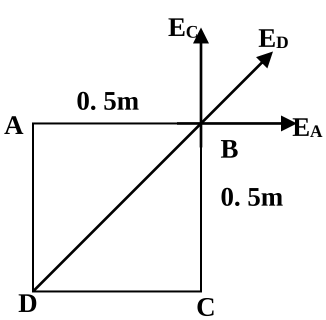
$$\vec{E} = k\frac{QQ_0}{Q_0 r^2} = k\frac{Q}{r^2}$$

Hence electric field intensity, due to charge, Q is given by;

$$\boldsymbol{\vec{E} = k\frac{Q}{r^2} = \frac{Q}{4ðå_0 r^2}}$$

***Example***

Calculate the electric field intensity at one corner of a square 0.5m if the outer three corners are occupied by point charges of magnitude $+8.24\times10^{-4}C$ .




$$\text{BD} = \sqrt{(0.5)^2 + (0.5)^2} = 0.707\text{m}$$

$$\text{E}_\text{A} = \text{k}\frac{Q_A}{r_{AB}} = \frac{9\times10^9(8.4\times10^{-4})}{(0.5)^2}$$
$$\text{E}_\text{A} = 2.97\times10^7 NC^{-1}$$

$$\text{E}_\text{C} = \text{k}\frac{Q_C}{r_{CB}} = \frac{9\times10^9(8.4\times10^{-4})}{(0.5)^2}$$
$$\text{E}_\text{C} = 2.97\times10^7 NC^{-1}$$

$$\text{E}_\text{D} = \text{k}\frac{Q_D}{r_{DB}} = \frac{9\times10^9(8.4\times10^{-4})}{(0.707)^2}$$
$$\text{E}_\text{D} = 1.48\times10^7 NC^{-1}$$

Resolving Horizontally;

$$\text{E}_\text{x} = \text{E}_\text{A}\cos\theta + \text{E}_\text{C}\cos\theta + \text{E}_\text{D}\cos\theta$$
$$\text{E}_\text{x} = 2.97\times10^7\cos0 + 2.97\times10^7\cos90 + 1.48\times10^7\cos45$$
$$\text{E}_\text{x} = 4.02\times10^7\text{NC}^{-1}$$

Resolving Vertically;

$$\text{E}_\text{y} = \text{E}_\text{A}\sin\theta + \text{E}_\text{C}\sin\theta + \text{E}_\text{D}\sin\theta$$
$$\text{E}_\text{y} = 2.97\times10^7\sin0 + 2.97\times10^7\sin90 + 1.48\times10^7\sin45$$
$$\text{E}_\text{y} = 4.02\times10^7\text{NC}^{-1}$$

Resultant Electric field intensity at B:

$$\text{E} = \sqrt{(E_x)^2 + \left(E_y\right)^2}$$
$$\text{E} = \sqrt{(4.02\times10^7)^2 + (4.02\times10^7)^2}$$
$$\mathbf{E = 5.67\times10^7NC^{-1}}$$

Direction of resultant force

$$\tan\theta = \frac{(E_x)}{\left(E_y\right)}$$
$$\tan\theta = \frac{4.02\times10^7}{4.02\times10^7}$$
$$è = \mathbf{45^0}$$

**Alternatively; Using vector form**

$$\vec{E} = \begin{pmatrix}\text{E}_\text{x}\\ \text{E}_\text{y}\end{pmatrix} = \begin{pmatrix}2.97\times10^7\cos0\\ 2.97\times10^7\sin0\end{pmatrix} + \begin{pmatrix}2.97\times10^7\cos90\\ 2.97\times10^7\sin90\end{pmatrix} + \begin{pmatrix}1.48\times10^7\cos90\\ 1.48\times10^7\sin90\end{pmatrix}$$

$$\vec{E} = \begin{pmatrix}\text{E}_\text{x}\\ \text{E}_\text{y}\end{pmatrix} = \begin{pmatrix}2.97\times10^7\\ 0\end{pmatrix} + \begin{pmatrix}0\\ 2.97\times10^7\end{pmatrix} + \begin{pmatrix}1.05\times10^7\\ 1.05\times10^7\end{pmatrix}$$
$$\vec{E} = \begin{pmatrix}\text{E}_\text{x}\\ \text{E}_\text{y}\end{pmatrix} = \begin{pmatrix}4.02\times10^7\\ 4.02\times10^7\end{pmatrix}$$

Thus the magnitude of the resultant intensity is;

$$\text{E} = \sqrt{(E_x)^2 + \left(E_y\right)^2}$$
$$\text{E} = \sqrt{(4.02\times10^7)^2 + (4.02\times10^7)^2}$$
$$\mathbf{E = 5.67\times10^7NC^{-1}}$$

Direction of resultant force

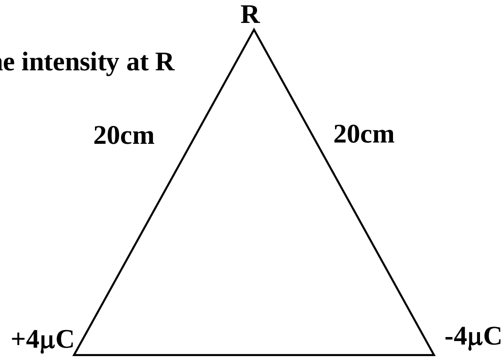
$$\tan\theta = \frac{(E_x)}{\left(E_y\right)}$$
$$\tan\theta = \frac{4.02\times10^7}{4.02\times10^7}$$

$\theta = \mathbf{45^0}$

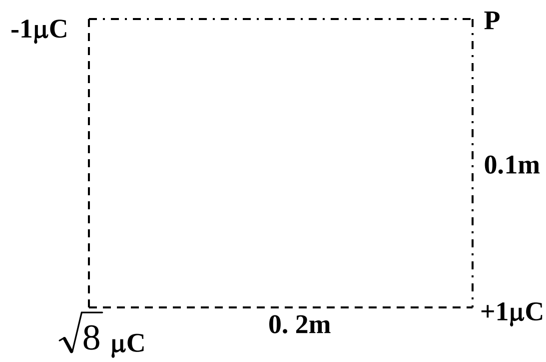
**EXERCISE**

1.

**Find the intensity at R**

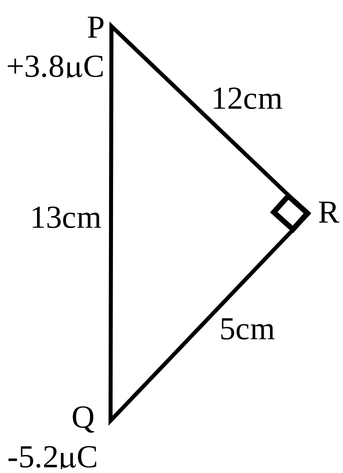



2.




Find the intensity at P

3. Two points +3.8μC and -5.2μC are placed in air at points P and Q as shown. Determine the electric field intensity at R.



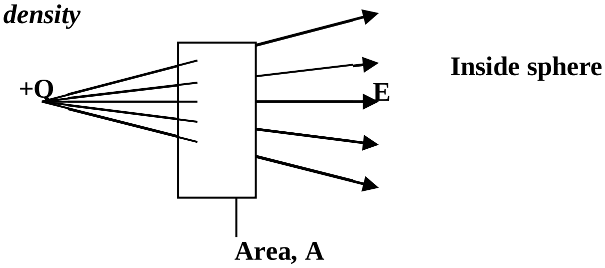

**Electric flux**

Electric field lines can be described by lines by force. The density of lines of force increases near the charge where the intensity is high.

The field strength or intensity, E at a point can be represented by the number of lines per unit area or flux density, through the surface perpendicular to the lines of force at the point considered.

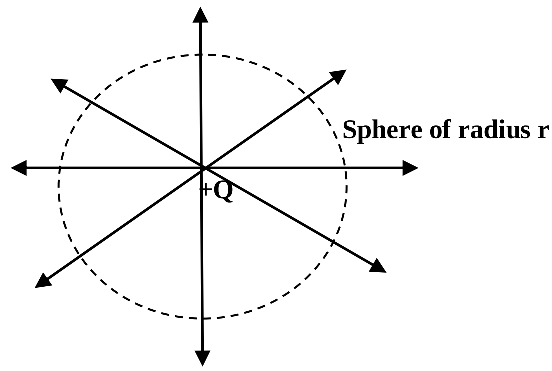
The *flux* through an area perpendicular to the lines of force *is the product of the electric field intensity and the area.*

***Relationship between electric field intensity and charge density***




Charge density is charge per unit area of the surface of the conductor.

Consider a sphere of radius r, drawn in space concentric with a point charge




**Total normal flux,ϕ = EA**

Let Q be the charge on the surface of the sphere and $\sigma$ be the charge density.(Charge per unit area).

$$Charge\ density = \frac{Charge}{Area}$$

$$\sigma = \frac{Q}{4\pi r^2}$$

$$Q = \sigma \times 4\pi r^2 \ldots\ldots\ldots.(i)$$

$$\vec{E} = \frac{Q}{\varepsilon_0 \times 4\pi r^2} \quad \ldots\ldots\ldots \text{(ii)}$$

Substituting for Q from Eqn (i) in equation (ii)

$$\vec{E} = \frac{\sigma \times 4\pi r^2}{\varepsilon_0 \times 4\pi r^2}$$

$$\boldsymbol{\vec{E}} = \frac{\sigma}{\varepsilon_{\mathbf{0}}}$$

Flux across the sphere.

Electric flux, $\phi = \vec{E} \times \text{Area}$

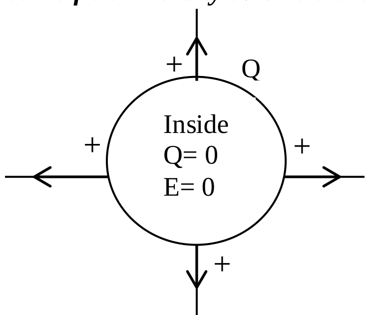
$$\phi = \frac{Q}{\varepsilon_0 \times 4\pi r^2} \times 4\pi r^2$$

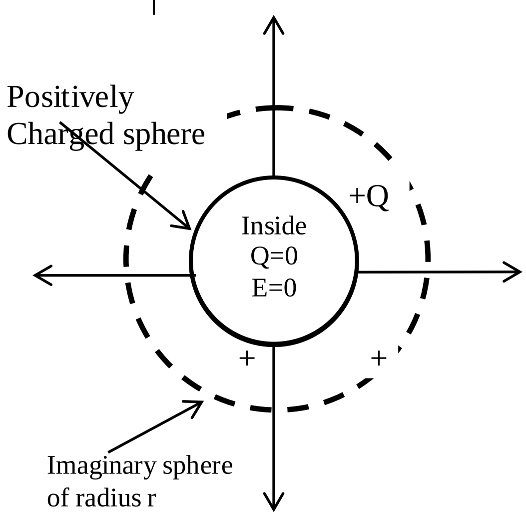
Electric flux, $\boldsymbol{\phi} = \frac{\boldsymbol{Q}}{\varepsilon_{\mathbf{0}}}$

The total flux crossing normally any sphere outside and concentrically around a point charge is constant. In general, the total flux passing normally through any closed surface whatever its shape is always equal to; $\frac{Q}{\varepsilon}$ where Q is the total charge enclosed by the surface and $\varepsilon$ is the permittivity of medium. This relation is called ***Gauss' theorem.***

**Application of Gauss' theorem**

***Electric field intensity outside a charged sphere***




The flux across a spherical surface of radius ***r***, concentric with a small sphere carrying a charge ***Q*** is given by, flux , $\phi = \frac{Q}{\varepsilon_0}$

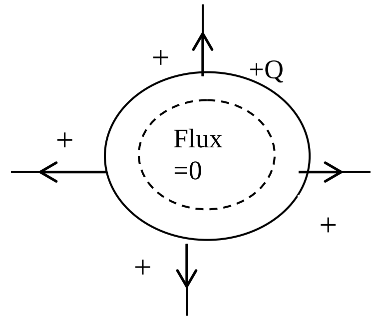
$$\Leftrightarrow, EA = \frac{Q}{\varepsilon_0}, \ \ But\ A = 4\pi r^2$$

$$E(4\pi r^2) = \frac{Q}{\varepsilon_0}$$

$$\boldsymbol{E = \frac{Q}{4\pi\varepsilon_0 r^2}}$$

The above expression is similar to that of a point charge. This means that outside a charged sphere, the field behaves as if all the charge on the sphere were concentrated at the centre.

**Electric field intensity inside a charged surface**

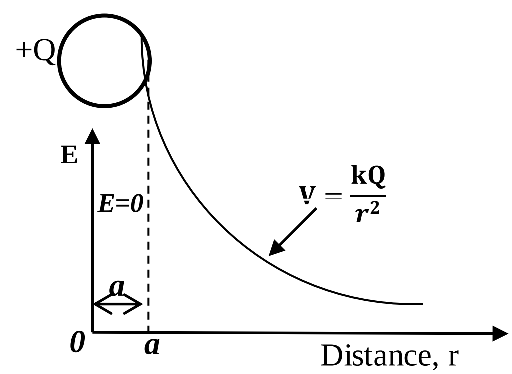


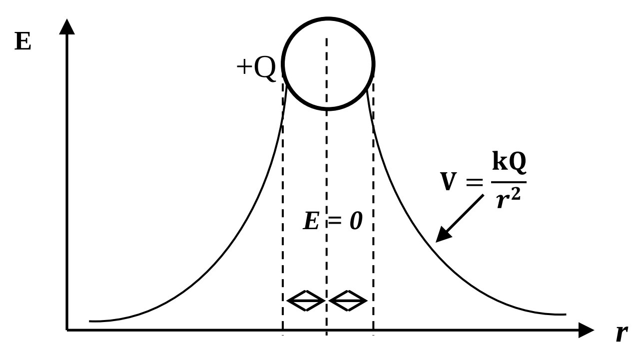

There is no charge inside the sphere hence flux = $\frac{Q}{\varepsilon} = \frac{0}{\varepsilon}$

$\therefore$ flux $= 0$; hence intensity, $E = 0$

Therefore, inside a charged body electric field intensity is zero.

**Variation of E with distance r from the centre of sphere**




***Example***

An isolated conducting spherical shell of radius **10cm** in a vacuum carries a charge of **+0.18μC**.

Calculate the electric field intensity at;

(i). The surface of sphere.
(ii). A point placed outside the sphere at a distance of 5cm from the sphere surface.
(iii). A point 5cm from the centre of the sphere

<u>Solution</u>

i. Radius of the sphere

$$E=\frac{Q}{4\pi\varepsilon_o r^2}=\frac{0.18\times9\times10^9\times10^{-6}}{\left(10/100\right)^2}=1.62\times10^5\,NC^{-1}$$

ii. Radius of imaginary sphere, r = 15cm = 0.15cm

$$E=\frac{Q}{4\pi\varepsilon_o r^2}=\frac{0.18\times10^{-6}\times9\times10^9}{0.15^2}=7.2\times10^4\,Nc^{-1}$$

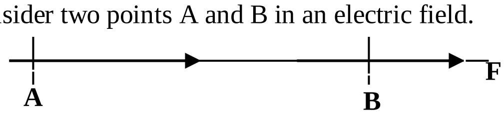
iii. A point 5cm from centre of the sphere.

Inside a sphere, flux is 0, so electric field is zero

## ELECTRIC POTENTIAL

Electric potential is the work done to move one coulomb of a charge from infinity to a point in an electric field.
Potential is a scalar quantity. The unit of potential is the volt (V) which is equal to a joule per coulomb. $(\mathrm{JC^{-1}})$

**Potential difference (pd)**
Consider two points A and B in an electric field.




If one moves one coulomb of a positive charge from B to A, work is done against the field and this work done is called the potential difference. Therefore the ***p.d between two points is thework done to move one coulomb of a positive charge from one point to another***. The unit of p.d is the volt (V).

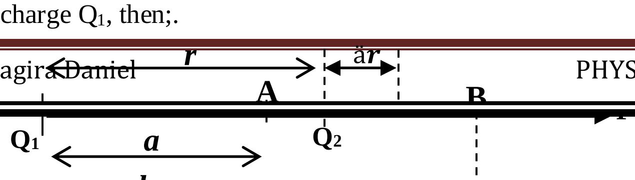
***Definition of a Volt*:** One volt is the p.d between two points is when the work done to move one coulomb of a positive charge from one point to another is one joule.

$$\therefore \text{P.d, V}=\frac{\text{work done (w)}}{\text{charge (Q)}};\ \text{V}=\frac{\text{W}}{\text{Q}}$$

1 volt = $1\mathrm{JC^{-1}}$

**Electric potential due to a point charge**
When a body is moved in afield, work is done. If it is moved in the direction of the field, then the field does the work. Otherwise, an external source of energy is required to move the body against the force field.

Consider two points A and B in an electric field of a single positive charge **+$Q_1$**in free space, at distances **a** and **b** respectively. When a charge, **$Q_2$** is placed at a distance **r** from the charge $Q_1$, then;.




$$\boldsymbol{Work\ done = Force\times displacement}$$
$$\ddot{a}W = F\ddot{a}r$$
$$\mathrm{W}=\int_a^b \mathrm{F.dr}\ ;\mathrm{Where, F}=\mathrm{k}\frac{\mathrm{Q_1Q_2}}{\mathrm{r^2}}$$
$$=\int_a^b \mathrm{k}\frac{\mathrm{Q_1Q_2}}{\mathrm{r^2}}.\mathrm{dr}$$
$$=\mathrm{kQ_1Q_2}\int_a^b\frac{1}{\mathrm{r^2}}.\mathrm{dr}$$
$$=\mathrm{kQ_1Q_2}\left[\frac{-1}{\mathrm{r}}\right]_a^b$$
$$\mathrm{W}=-\mathrm{kQ_1Q_2}\left(\frac{1}{\mathrm{b}}-\frac{1}{\mathrm{a}}\right)$$

$$\mathbf{W}=\mathbf{kQ_1Q_2}\left(\frac{\mathbf{1}}{\mathbf{a}}-\frac{\mathbf{1}}{\mathbf{b}}\right)$$

- ❖ This is the value of the work which must be done externally to carry a positive charge $Q_2$ from B to A
- ❖ But work done = potential energy. Hence Potential energy = **$(kQQ_0)/r$.**
- ❖ The work done per Coulomb is the Electric Potential difference $V_{AB}$ between points A and B.

$$\mathbf{V_{AB}}=\frac{\mathbf{W}}{\mathbf{Q_2}}=\mathbf{kQ_1}\left(\frac{\mathbf{1}}{\mathbf{a}}-\frac{\mathbf{1}}{\mathbf{b}}\right)$$

If point B is very far away from point A, i.e as $b\to\infty,\frac{1}{b}\to0$,

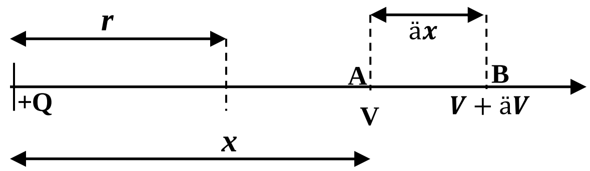
$$\mathrm{V_{AB}}=\frac{\mathrm{W}}{\mathrm{Q_2}}=\mathrm{kQ_1}\left(\frac{1}{\mathrm{a}}\right);\mathrm{where, a = r.}$$

$$\mathbf{V_{AB}}=\frac{\mathbf{kQ_1}}{\boldsymbol{r}}$$

In electrostatics, when a positive charge moves in the direction of the field, the electric potential energy lost by the field-charge system can be regained if the charge is moved by an external agent in the opposite direction.

In current electricity, however, the energy lost by the electric field, inside the electric conductor is irrecoverable since the heat produced cannot be converted back to other forms of energy by reversing current.

**Relationship between E and V(P.d).**
Consider two points A and B which are in an electric field at a distance***x***and a distance **($x+\Delta x$)** from charge Q respectively.




If **V** is the potential at A and **(V+ΔV)** is the potential at B, then the P.d between A and B.

$V_{AB} = V - (V + \delta V)$

$V_{AB} = -\delta V$………………………………. (i)

The work done in moving a charge Q from B to A is given by;

$W = E.Q \times \text{ä}\boldsymbol{x}$………………………… (ii)

The work done in moving a charge Q through a potential difference is given by;

$W = QV_{AB}$…………………………… (iii)

From equations (ii) and (iii);

$E.Q \times \text{ä}\mathbf{x} = -\delta V$

$$E.Q = \frac{-\delta V}{\text{ä}\mathbf{x}}$$

For 1C of charge Q, Q=1C.

$$E. = -\frac{\delta V}{\delta x}$$

For a non uniform field;

$$\mathbf{E.} = -\lim_{\text{äx}\to 0}\frac{\text{ä}\mathbf{V}}{\text{ä}\boldsymbol{x}} = -\frac{\mathbf{dV}}{\mathbf{d}\boldsymbol{x}}$$

The negative sign means that the distance increases the potential drop.

For a uniform field;

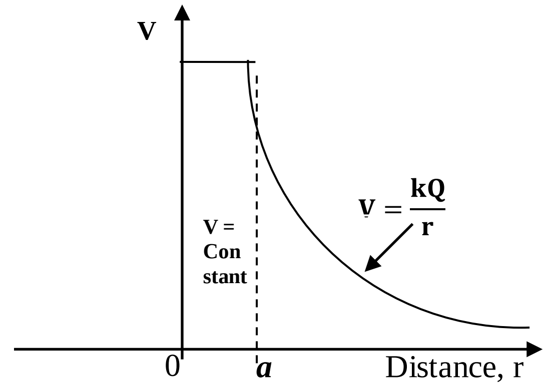
$$\mathbf{E.} = \frac{\mathbf{V}}{\mathbf{d}}$$

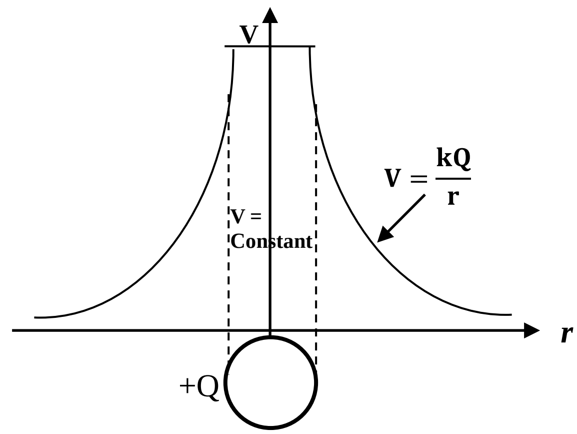
$$\textbf{Electric field intensity.} = \frac{\textbf{Change in in potential}}{\textbf{distance}}$$

Thus another unit of electric field intensity is a volt per metre($\boldsymbol{Vm^{-1}}$).

The value $\frac{\text{äV}}{\text{äx}}$ is called the potential gradient.

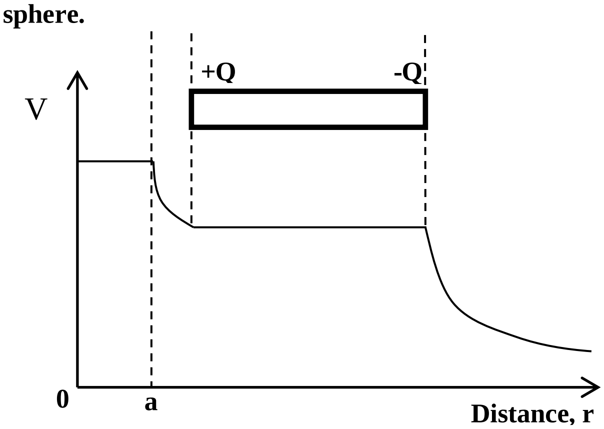
**Variation of V with distance r from the centre of sphere**



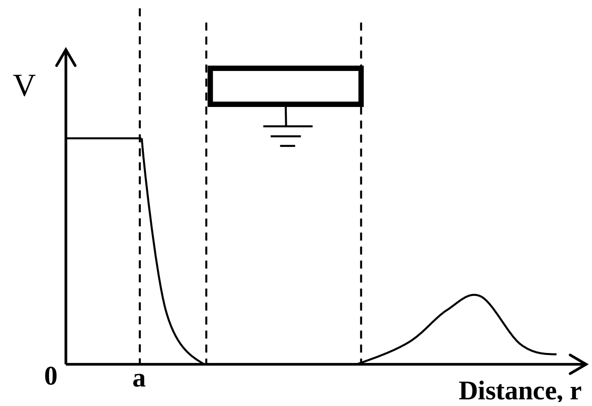




NOTE:
1. The electric field intensity E inside a charged sphere is zero. Since there is no charge inside the sphere.
2. The electric potential, V inside a charged sphere is constant.

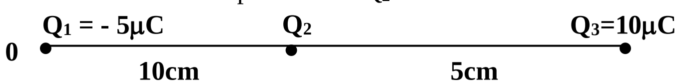
**Variation of V relative to the earth along the axis of a metal rod with distance r from the centre of a charged sphere.**




When the rod is earthed, it comes to the potential of the earth. i.e zero potential.




**Example:1**
Calculate the electric potential at $Q_2$.




Electric potential at $Q_2$ du to $Q_1$

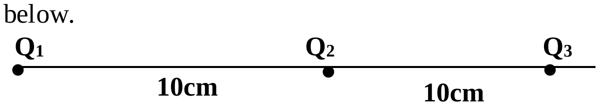
$$V_1 = \frac{kQ_1}{r} = \frac{9.0 \times 10^9(-5 \times 10^{-6})}{10 \times 10^{-2}} = -4.5 \times 10^5 V$$

Electric potential at $Q_2$ du to $Q_3$

$$V_3 = \frac{kQ_3}{r} = \frac{9.0 \times 10^9(10 \times 10^{-6})}{5 \times 10^{-2}} = 1.8 \times 10^5 V$$

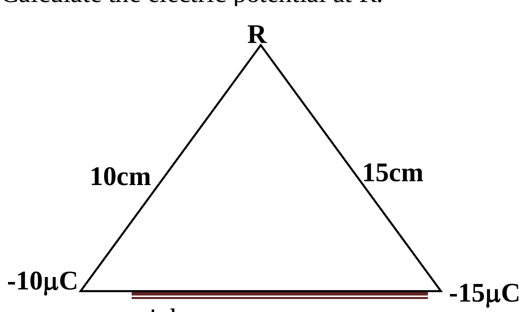
Electric potential at $Q_2$:
$V_2 = V_1 + V_3$
$V_2 = -4.5 \times 10^5 + 1.8 \times 10^5$
$\underline{\underline{V_2 = 1.35 \times 10^6 V}}$

**Example:2**
Three point charges $Q_1$, $Q_2$ , $Q_3$ of magnitude **+5μC, 6μC, and -20μC** respectively are situated long a straight line as shown below.




Calculate the;
**(i)** Electric field intensity mid-way between **$Q_1$** and **$Q_2$ .** [Ans: $E_m = 4.4 \times 10^6 NC^{-1}$]
(ii) Electric potential mid-way between $Q_1$ and $Q_2$ .
[Ans: $V_m = 7.8 \times 10^5 V$]

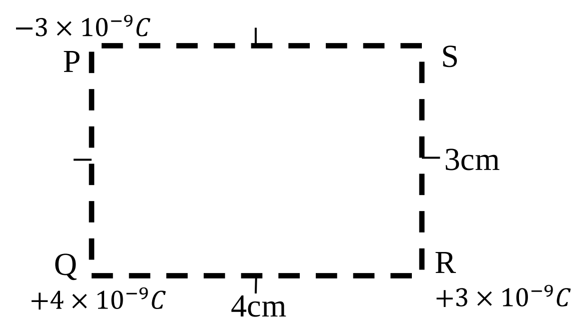
**Example:3**
Calculate the electric potential at R.




**Example:4**
Two points A and B are 15cm horizontally and 20cm vertically from a **6.0μC** charge. Find the;
(i) Electric potential difference between A and B
(ii) Energy required to bring a charge of **1μC** from infinity to point A.
(iii) What is the significance of the sign of the energy in (ii) above?

**Example:4**
Three point charges $Q_1$, $Q_2$, $Q_3$ and $Q_4$ of magnitude **-1μC, +2μC, -3μC and +4μC** respectively are situated long a straight line in that order such that they're 20cm from each other.
(i) Calculate the potential energy of $Q_2$.
(ii) What is the significance of the energy in (i) above?

**Example:5**
Three charges of $-3 \times 10^{-9}C, +4 \times 10^{-9}C$ $and +$ $3 \times 10^{-9}C$ are placed in a vacuum at the vertices PQR of a rectangle respectively of sides 3cm by 4cm s shown below.




Calculate the;
(i) Resultant field intensity at S. [$Ans: 39009.3NC^{-1}\ t\ 82.1^0$]
(ii) Work done in moving a $+4 \times 10^{-9}C$ charge from Q to S. [$A\ ns$: 720J]

**Example:6**
An á $- pearticle$ of charge $+3.2 \times 10^{-19}C$ and mass $6.8 \times 10^{-27}kg$ is travelling with a velocity of $1.0 \times 10^{-7}ms^{-1}$ directly towards nitrogen nucleus and which has a charge of $11.2\times 10^{-19}C$. Calculate the closest distance of approach of the á $- pearticle$ to the nitrogen nucleus.
(Assume that the á $- pearticle$is initillyat a very long distance from N).
[Ans: $r = 9.487 \times 10^{-15}m$]

**EQUIPOTENTIAL SURFACES**
These are surfaces which have got the same (or constant) potential through out. The potential difference between any two points on an equipotential surface is zero.
Equi potential surfaces are always at right angles to the lines of force.

**Characteristics of Equi-potential surfaces.**

(i). All points on the surface are at the same potential. The p.d btn any two points on the surface is zero.
(ii). Energy required to move a charge from one point to another is zero.
(iii). Electric field lines from any charge are perpendicular to the surface.
(iv). They are conductors in electrostatic equilibrium.

**Why equi-potential surfaces are perpendicular to the electric field lines**
All points on an equi-potential surface are at the same potential. Thus the p.d between any two points is zero, implying that the work done in moving a charge from one point on the surface to another is zero.

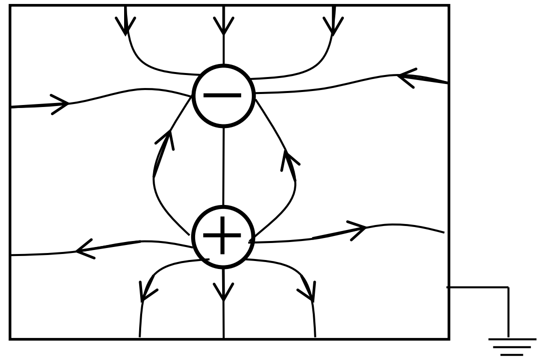
Therefore, electric field lines must be perpendicular such that the component of field intensity along the surface is zero ($\vec{E}\cos 90$) Otherwise some work would be done.
**Examples of equi-potential surfaces**
(i). Point charge
When electric field lines are sketched around an isolated positive charge, the different radial distances from the charge mark out an equi-potential surface.

(ii). Metal conductor
A conductor in electrostatic equilibrium is an equi potential surface. This is because, inside the charge inside a conductor and those outside are the equa and opposite. Therefore they cancel out at every point of the conductor leaving the conductor with the same potential through out the surface of the conductor.

(iii). Earth
It is ta very large conductor that any charge in the atmosphere can not raise its potential appreciably.
All points on the earth are approximately at the same potential. The potential difference between any two points on the earth is zero. Thus the earth is said to be at zero potential.

**ELECTROSTATIC SHIELDING**
Is the process of protecting a body from intense electric fields which may cause shock by enclosing it inside a conductor.
It is the creation of a region free from effects of electric fields.

The fact that there is no electric field inside a closed conductor when it contains no charged bodies is applied in the Faradays' cage used in high voltage measurements to protect people and instruments from the intense electric field shocks.



All points in the metal cage (box) are at the same potential. Therefore there is no P.d between any two points inside the cage. (p.d = 0).Thus the work done to transfer a charge from one point to another is zero (0J).
This means that the component of force due to electric field along the surface of the conductor is zero. This explains why the electric field lines should be perpendicular to the surface.
The charged body inside the cage (box) is thus said to be protected from the intense electric shock due to the intense electric field.

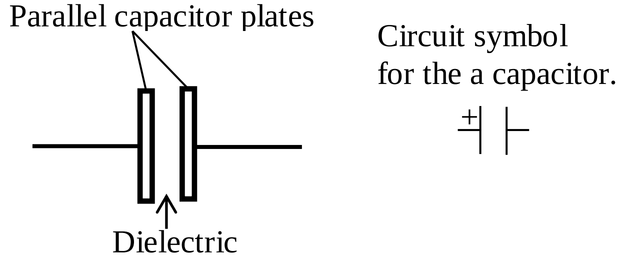
**NOTE: The cage should NOT be earthed when used to shield any thing inside it.**

# CAPACITORS

**A capacitor** is a device used for storing charge or electrical energy.
Basically consists of two conductors, such as a pair of parallel metal plates of equal dimensions separated by an insulator called a dielectric.

**A dielectric** is an insulating medium that is sandwiched by the plates of a capacitor.

The dielectrics used include: Air (e.g in variable capacitors used in tuning of radios to selsct frequencies), mica (e.g in multiple capacitors), Alluminium oxide (e.g in electrolytic capacitors), Plastic, Oil, paper, paraffin wax, ceramics, e.t.c.



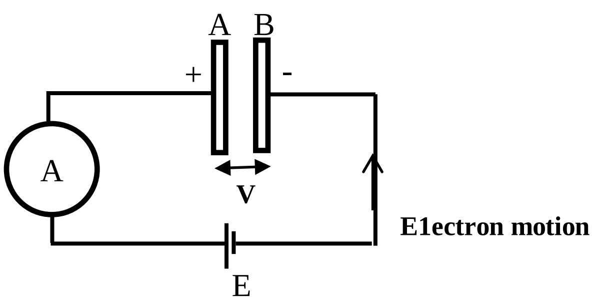

**Applications of capacitors.**
Because of its ability to store charge, capacitors have many uses. These include;

- Tuning (selecting frequencies in) radios
- Elimination of sparking in circuits
- Smoothening rectified currents
- Photographic flash units
- Timing circuits.

**Charging and discharging of a capacitor**

**(i). Charging process**

Capacitor plates A and B are connected to a d.c. supply.



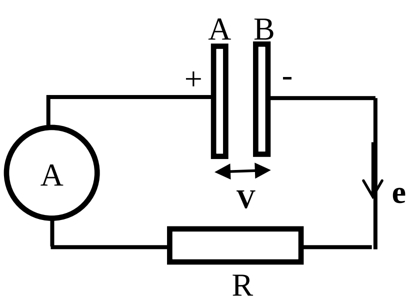

Electrons move from the negative terminal to plate B making it negatively charged. Electrons from plate B repel an equal number of electrons from plate A which then becomes positively. In this case, the capacitor is said to be charging. The charging current is given by; $\mathbf{I = \frac{E-V}{r}}$

The charging process continues until the p.d, V across the capacitor is equal to the p.d, E of the battery.

When charging has just began, there is no charge on C and hence no p.d across it (V=0). Thus the initial current flowing is $\mathbf{I_{max} = \frac{E}{r}}$, and the ammeter indicates maximum deflection which gradually decreases to zero as charging continues.

When the charging process is complete, the ammeter shows no deflection, (i.e.I=0). At this stage, the capacitor is fully charged.

**(ii). Discharging process**




When the capacitor is fully charged, the battery is removed and replaced by a load (resistor, R).

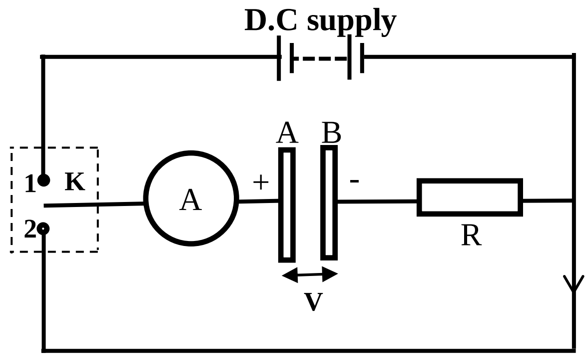
Electrons move from plate, B to plate A until the p.d, V across the plates is zero.

At the beginning of discharging, the ammeter indicates maximum deflection which gradually decreases to zero as discharging continues.

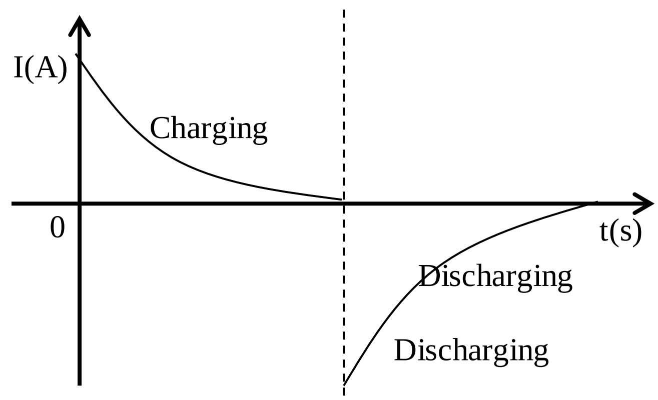
When discharging is complete, the ammeter shows no deflection, (i.e, I=0).

**NOTE:**

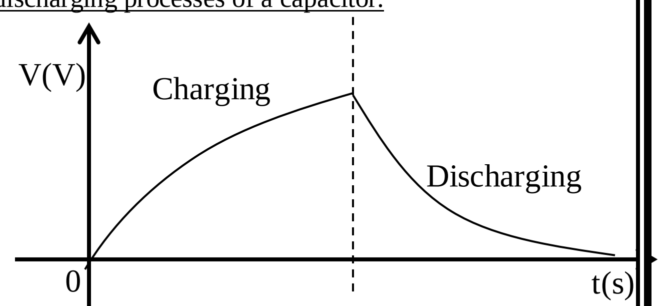
The two processes can also be investigated using the circuit below.




<u>A graph of Current, I against time for charging and discharging processes of a capacitor.</u>



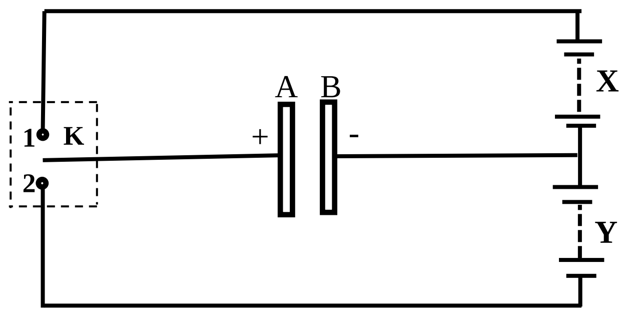

<u>A graph of potential difference, V against time for charging and discharging processes of a capacitor.</u>




Generally, a capacitor is charged when a battery of p.d is connected to it.

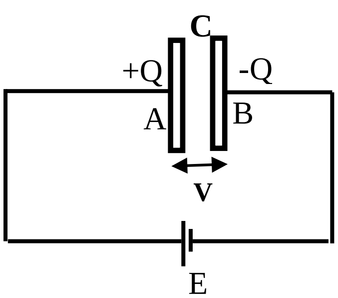
When the plates of the capacitor are joined together, the capacitor becomes discharged (loses the charge on the plates)

**Flow of a.c through a capacitor**
A.c can flow through a capacitor unlike d.c.




When contact is made at 1, current flows from the battery X and charges plate A positively.
When the contact is made at 2, current flows from battery Y and discharges plates A. Plate, A loses its positive charge and becomes negatively charged.
If the switch is rocked between 1 and 2, continuously, current flows backwards and forwards along the wires connected to the capacitor.
An a.c voltage reverses many times in a second, therefore an a.c can flow in the connecting wires of a capacitor.

**CAPACITANCE OF A CAPACITOR (C)**
Consider a capacitor of capacitance C in series with a battery as shown below.




The magnitude of charge developed on plate A is equal to the magnitude of charge developed on plate B.
(i.e. $|Q_A| = |Q_B| = Q$.
The charge Q developed on either capacitor is proportional to the P.d, V across the capacitor plates.
i.e, $Q$ á $V \Leftrightarrow \boldsymbol{Q = CV}$
$Thus;\ \boldsymbol{Q = CV}$
The constant C is called the capacitance of a capacitor.

**Capacitance of a capacitor** is the ratio of the magnitude of the charge on either plate to the p.d. between the plates of the capacitor.

$$\text{Capacitance, C} = \frac{\text{Charge, Q on a plate}}{\text{Potential difference, V across the plates}}$$

$$\mathbf{C} = \frac{\boldsymbol{Q}}{\boldsymbol{V}}$$

The S.I unit of capacitance, C is the farad (F)

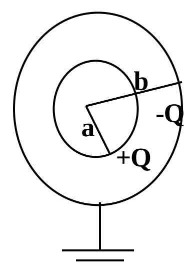
$1F = 1CV^{-1}$ $\quad$ $1\text{pF} = 10^{-12}\,F$

$1\mu F = 10^{-6}\,F$ $\quad$ $1\text{nF} = 10^{-9}\,F$

**A farad** is the capacitance of a capacitor when 1C of charge is developed on tits plates when the P.d across the plates is 1V.

**Capacitance of an isolated charged sphere**
The potential V of a sphere of radius $\boldsymbol{r}$, in a medium of permittivity, ε, and having a charge Q is; $V = \frac{kQ}{r} = \frac{Q}{4\text{ðå}r}$.
From;$\text{C} = \frac{Q}{V} = Q \times \frac{4\text{ðå}r}{Q} = 4\text{ðå}r$
$\mathbf{C = 4}\text{ðå}\boldsymbol{r}$
For free space, $\text{C} = 4\text{ðå}_0 r$.

**Capacitance of concentric spheres**
Consider two concentric sphere of radii a and b




Let $Q^+$ be the charge on the inner sphere and the outer sphere be earthed with air between them. The induced charge on the outer sphere is $^-$Q. the potential
$V_a$ of the inner sphere = Potential due to $^+$Q + potential due to $^-$Q.

$V_a = k\frac{+Q}{a} + k\frac{-Q}{b}$.

$V_a = \frac{kQ}{a} - \frac{kQ}{b} \Leftrightarrow .V_a = kQ\left(\frac{1}{a} - \frac{1}{b}\right) \Leftrightarrow V_a = kQ\left(\frac{b-a}{ab}\right)$

$$V_a = \frac{kQ(b-a)}{ab} = \frac{Q(b-a)}{4\text{ðå}_0(ab)}$$

Potential of outside sphere, $V_b = 0$ (since it is earthed, there is no charge). Therefore the p.d between the inner and outer sphere is given by;
$V = V_a - V_b$

$$V = \frac{kQ(b-a)}{(ab)} - 0 \Leftrightarrow V = \frac{kQ(b-a)}{(ab)}$$

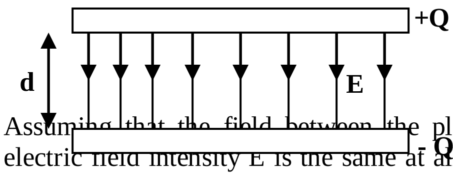
Hence From the definition of capacitance; capacitance, C is;

$$C = \frac{Q}{V} = \frac{Q}{\frac{kQ(b-a)}{(ab)}} = \frac{(ab)}{k(b-a)} = \frac{4\text{ðå}_0(ab)}{k(b-a)}$$

$$\boldsymbol{C} = \frac{\boldsymbol{(ab)}}{\boldsymbol{k(b-a)}} = \frac{\mathbf{4}\text{ðå}_\mathbf{0}\boldsymbol{(ab)}}{\boldsymbol{k(b-a)}}$$

**Capacitance of a parallel plate capacitor**

Consider a capacitor with plates of common area A, separated by a medium of thickness d, and permittivity, ε. Of one plate has a charge +Q, the other with –Q.




Assuming that the field between the plates is uniform, the electric field intensity E is the same at all points and is given by ;

From Gauss's law;

$\vec{E} = \frac{ó}{å} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(i)$

From definition of charge density; ó:

$ó = \frac{Q}{A} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(ii)$

Substituting for ó in Equation (i)

$\vec{E} = \frac{Q}{åA} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(iii)$

But $\vec{E}$ between plates is equal to the potential gradient between the plates. i.e;

$\vec{E} = \frac{V}{d} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots...(iv)$

Equating equation (iii) and (iv) gives;

$\frac{V}{d} = \frac{Q}{åA} \Leftrightarrow \frac{Q}{V} = \frac{åA}{d}$

But $\frac{Q}{V} = C$: Hence the capacitance, C of the capacitor is;

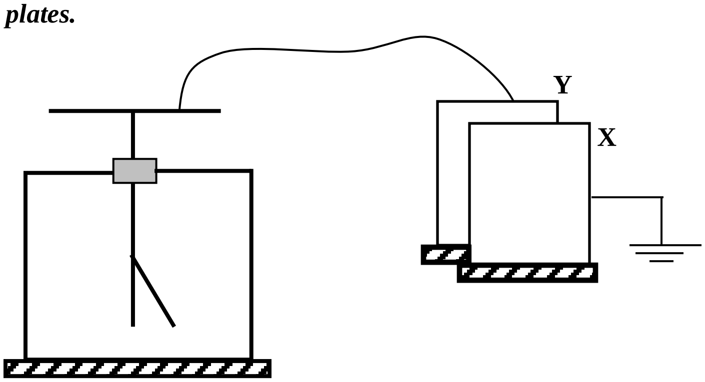
$$\mathbf{C = \frac{åA}{d}}$$

If the medium between the plates is air, then; $å = å_0$, Hence

$$\mathbf{C = \frac{å_0 A}{d}}$$

**Experiment to investigate the factors affecting the capacitance of a parallel plate capacitor.**

***(i) Dependence of capacitance on separation, d, between plates.***




Consider two plates X and Y with X earthed and Y connected to an insulated gold leaf electroscope.

Plates X and Y are set close to each other but not touching. Plate Y is given a charge Q by means of an electrophorus. The divergence of the leaf of the electroscope is noted and it is a measure of the p.d between X and Y.

Plate X is moved nearer to Y, without touching and the divergence of the leaf is observed to decrease meaning that the p.d between X and Y has decreased.

Hence capacitance, C, increases. Hence decrease in separation between the plates of the capacitor increases with capacitance of the capacitor.$\Longrightarrow \boldsymbol{C}$ á $\frac{\mathbf{1}}{\boldsymbol{d}}$……………..…(i)

***(ii) Dependence of capacitance on area, A of overlap of plates.***

Using the set in (i) plate Y is given a charge and the divergence of the leaf of electroscope noted. Plate X is then carefully displaced vertically downwards keeping separation, d, constant. The leaf of the electroscope is observed to increase meaning that the p.d between the plates has increased. Hence capacitance decreases.

By displacing plate X vertically downwards, the area of the plates decreases. Hence capacitance decreases with decrease of area. .$\Longrightarrow \boldsymbol{C}$ á $\boldsymbol{A}$……………………(ii)

*(Note: Area can be increased by joining another identical pair of metal plates to the previous pair of plates)*

***(iii) Dependence of capacitance on the nature of the medium between the plates.***

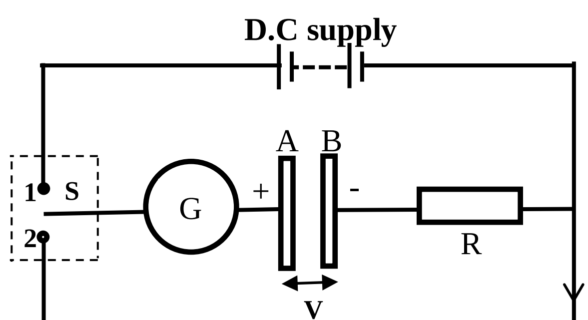
Using the apparatus in step (i), the common area and separation are kept constant. Plate Y is again charged and the divergence of the leaf noted. A sheet of glass is placed between the plates X and Y. The leaf divergence is observed to decrease. This means that p.d between X and Y has decreased. Hence the capacitance,C increases.

Therefore when a dielectric (insulator) is placed between plates of a capacitor, the capacitance of the capacitor increases. .$\Longrightarrow \boldsymbol{C}$ á å………………..(iii)

Thus from equations (i), (ii) and (iii);

.$\Longrightarrow \boldsymbol{C}$ á $\frac{\boldsymbol{åA}}{\boldsymbol{d}}$

**The factors can also be investigated using a a reed switch and a light sensitive galvanometer.**



The capacitor, C is alternately charged and discharged through a light sensitive galvanometer (or a micro- ammeter) at a frequency, f using a reed switch, S (vibrator) energized by a low a.c. of frequency, f, from the mains supply.
The charge, Q developed on the capacitor on charging is proportional to the current, I through the galvanometer, since;
$\boldsymbol{I = fQ}$**…………..(i)**
But also, for a given P.d, V, the charge Q is proportional to the capacitance, C. Thus,
$\boldsymbol{Q = CV = C(IR)}$**…………..(i)**
From (i) and (ii);
$\boldsymbol{I = (fV)C \Rightarrow C \propto I}$

- Keeping the charging p.d V and area of overlap, A constant, the current, I flowing for different plate separations, d is observed and noted.

The results show that, $\boldsymbol{I \propto \frac{1}{d} \Rightarrow C \propto \frac{1}{d}}$

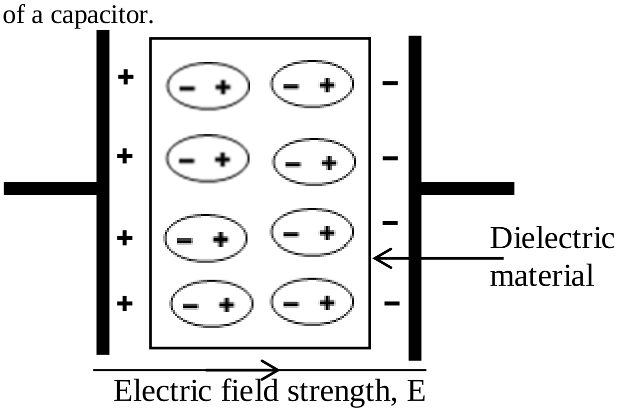
- Keeping the charging p.d V and separations, d constant, the current, I flowing for different plate areas of overlap, A is observed and noted.

The results show that, $\boldsymbol{I \propto A \Rightarrow C \propto A}$

- Keeping the charging p.d V, separations, d and area of overlap, A constant, the current, I flowing when a dielectric is sandwiched by the capacitor plates is observed and noted.

The results show that, $\boldsymbol{I \propto}$ å $\boldsymbol{\Rightarrow C \propto}$ å

**Effect of a dielectric on the capacitance of a capacitor**
A dielectric is an insulating material sandwiched by the plates of a capacitor.




When the dielectric is sandwiched between the plates of an isolated charged capacitor, the molecules of the dielectric get polarized by the electric field intensity between the plates.
At the surface of the dielectric, charges appear of opposite sign to those of the nearest (closest) plate. But since these are bound charges,, an electric field developes between the opposite faces of the dielectric in a direction opposite to the applied field. This reduces the resultant electric field intensity and hence the P.d between the plates (since $\boldsymbol{E = \frac{V}{d}}$**).**
Since $\boldsymbol{C = \frac{Q}{V}}$ **,** it implies that the capacitance, C, increases.

If the capacitor is connected to a battery, it's p.d, V,is constant but the surface charges of the dielectric, enable additional charges (ä$Q$) to be added on the plates by offsetting the charges on the plates. This means that charge Q increases, but $C = \frac{Q}{V}$. Therefore, the capacitance C increases

If initially, the plates have capacitance $C_0$ and charge$Q_0$, then on introducing the dielectric, the additional charges, ä$Q$, change the capacitance from $C_0$ to $C_1$.
$Q_0 = C_0V$ …………………..(i)
$(Q_0 + \text{ä}Q) = C_1V$ …………(ii)

Equation (ii) divide by equation (i) gives;

$$\frac{(Q_0 + \delta Q)}{Q_0} = \frac{C_1V}{C_0V} \Leftrightarrow \left(\mathbf{1} + \frac{\delta Q}{Q_0}\right) = \frac{C_1}{C_0}$$

$$\Leftrightarrow C_1 = \left(\mathbf{1} + \frac{\delta Q}{Q_0}\right)C_0$$

Hence the capacitance increases by$\frac{\delta Q}{Q_0}$on introducing a dielectric.

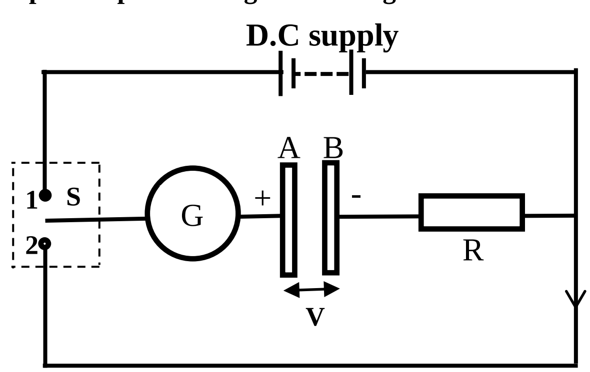
**<u>Dielectric Strength;</u>**
Is the maximum potential difference per metre thickness which a dielectric can withstand without breaking down.
OR;
It is the potential gradient at which the insulation of a dielectric breaks down and a spark passes through.

**<u>Relative permittivity, (dielectric constant)</u>**
Relative permittivity is the ratio of the capacitance of a capacitor with a dielectric between its plates to the capacitance of the same capacitor without a dielectric between the plates.

$$\text{å}_{\boldsymbol{r}} = \frac{\boldsymbol{C_m}}{\boldsymbol{C_0}} = \frac{\left(\text{å}_m A/_d\right)}{\left(\text{å}_0 A/_d\right)} = \frac{\text{å}_{\boldsymbol{m}}}{\text{å}_{\boldsymbol{0}}}$$

Thus relative permittivity is the ratio of the permittivity of a dielectric to the permittivity of free space.

**Measurement of Permittivity,** å **of the medium between the capacitor plates. using a Ballistic galvanometer.**




The area of overlap, A and the separation d of the capacitor plates are measured.

When the vibrating bar makes contact with 1,the capacitor, C is ,charged from the supply p.d, V measured on a voltmeter.
When the vibrating bar makes contact with 2, the capacitor, C discharged through a light sensitive galvanometer (or a micro-ammeter) at a frequency, f using a reed switch, S (vibrator) energized by a low a.c. of frequency, f, from the mains supply. This gives an average steady current I on the galvanometer
The charge, Q developed on the capacitor on charging is proportional to the current, I through the galvanometer, since;
$\mathbf{I = fQ}$ ………………....... **(i)**
But also, for a given P.d, V, the charge Q is proportional to the capacitance, C. Thus,
$Q = C\mathbf{V}$ **and** $\mathbf{C = \frac{åA}{d}}$

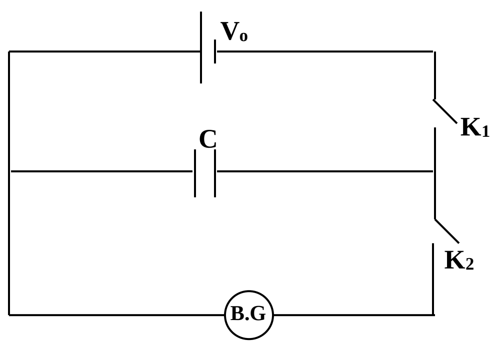
$$\mathbf{Q} = \left(\frac{å\mathbf{A}}{\mathbf{d}}\right)\mathbf{V} \dots\dots\dots\dots..(\mathbf{ii})$$

Thus from equations (i) and (ii), we have;

$$\mathrm{I} = \left(\frac{\varepsilon \mathrm{A}}{\mathrm{d}}\right)\mathrm{Vf}$$

$$å = \frac{\mathbf{Id}}{\mathbf{AVf}}$$

**Measurement of capacitance of a capacitor using a Ballistic galvanometer.**




The standard capacitor of known capacitor $C_1$ is placed in position of C. Switch $K_1$ is closed for some time while $K_2$ is open. It is then opened.
In this case the capacitor C1 charges to p.d Vo, where its charge, $Q_1 = C_1V_o$.
$K_2$ is then closed and the deflection, $\theta_1$, of the ballistic galvanometer noted. $Hence, Q_1 = kè_1$

$$Hence, C_1V = kè_1 \dots\dots\dots\dots\dots\dots\dots\dots(i)$$

The capacitor of unknown capacitance $C_2$ replaces $C_1$. Switch $K_1$ is closed for some time while $K_2$ is open. It is then opened.
In this case the capacitor $C_2$ charges to p.d Vo, where its charge, $Q_2 = C_2V_o$.
$K_2$ is then closed and the deflection, $\theta_2$, of the ballistic galvanometer noted. $Hence, Q_2 = kè_2$

$$Hence, C_2V = kè_2 \dots\dots\dots\dots\dots\dots\dots(ii)$$

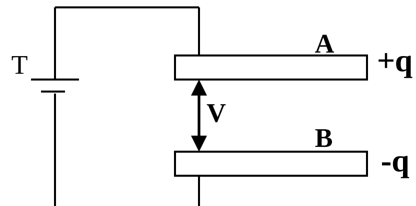
Hence equation (ii) divided by (i) gives;

$$\frac{\boldsymbol{C_2}}{\boldsymbol{C_1}} = \frac{\mathbf{è_2}}{\mathbf{è_1}} \Leftrightarrow \boldsymbol{C_2} = \frac{\mathbf{è_2}}{\mathbf{è_1}} \times \boldsymbol{C_1}$$

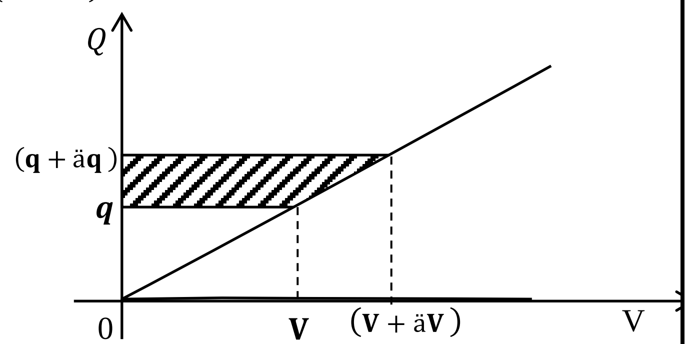
***Note:***

***This experiment can be used in comparison of capacitances of different capacitors.***

***Energy stored in a capacitor***
Consider a parallel plate capacitor of capacitance C being charged.




The charging process consists of transferring charge from plate A to plate B. Suppose that at some stage, during the charging process, the p.d between the plates is V when a small amount of charge $+ä\mathbf{q}$is transformed from plate B to plate A. The charge on A increases to$(\mathbf{q} + ä\mathbf{q})$and the p.d increases to $(\mathbf{V} + ä\mathbf{V})$.




The work done to transfer charge $+ä\mathbf{q}$from B to A is equal to the area of the shaded part.**;**
Total work done = Energy stored = Area of shaded part

$$\text{Area of a trapezium} = \frac{1}{2}(\mathbf{q} + ä\mathbf{q} - \mathbf{q})(\mathbf{V} + \mathbf{V} + ä\mathbf{V})$$

$$\text{Total work done} = \frac{1}{2}(ä\mathbf{q})(\mathbf{2V} + ä\mathbf{V})$$

$$\text{Total work done}, ä w = \mathrm{V}\delta \mathrm{q} + \frac{1}{2}\delta \mathrm{q}(\delta \mathrm{V})$$

But$+ä\mathbf{q}$ is much smaller than $q$ so even$\delta V \lll V$
Hence;$\delta V\delta q \rightarrow 0$
$\textbf{Total work done}, ä\boldsymbol{w} = \mathbf{V}ä\mathbf{q}$

$ä w = (\mathrm{V} + \delta\mathrm{V})\delta \mathrm{q}.$

$$ä w = \mathrm{V}\delta\mathrm{q} + \delta\mathrm{V}\delta\mathrm{q}$$

But$+ä\mathbf{q}$ is much smaller than $q$ so even$\delta V \lll V$
Hence;$\delta V\delta q \rightarrow 0$
Thus; $ä\boldsymbol{w} = \mathbf{V}ä\mathbf{q}$
But$V = \frac{q}{C}$

$$ä\boldsymbol{w} = \frac{\boldsymbol{q}}{\boldsymbol{C}}ä\mathbf{q}$$

To charge the capacitor plate A from q = 0 to q = Q, the work done by the battery is;

$$W = \int_0^Q \frac{q}{C}dq = \frac{Q^2}{2C}.\ But\ \ Q = CV$$

$$W = \frac{Q^2}{2C} = \frac{CV^2}{2} = \frac{Q^2}{2C}$$

**Alternatively;**
Suppose the final charge, Q moved from one plate to another through an average P.d of $\frac{(0+V)}{2}$. Since at start there is zero p.d and that at the end is V.

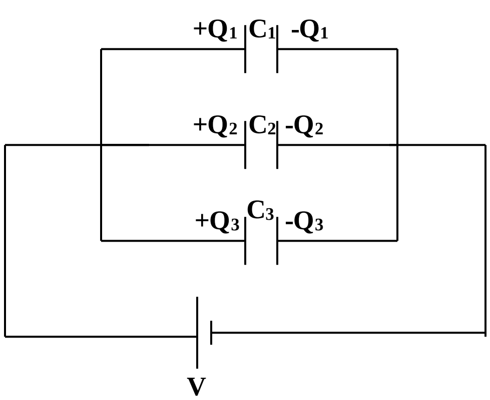
$$Total\ work\ done = Energy\ stored$$
$$= Average\ P.d \times Charge$$
$$= \frac{(0+V)}{2} \times Q$$
$$= \frac{1}{2}QV$$

**Note;** The charging battery supplies an amount of energy equal to QV during the charging process. Half of this energy is ($\frac{1}{2}QV$) is stored in the capacitor and the other half is transferred as heat in the resistive circuit.

The heat produced is independent of the circuit resistance. However a high resistance slows down the charging current and so delays the charging process while a low resistance enables the charging process to be faster.
In both cases, heat is produced.

**Arrangement of capacitors**
Electronic circuits like radios, T.Vs and computers circuits have capacitors which appear in different arrangements whose resultant capacitance must be known.

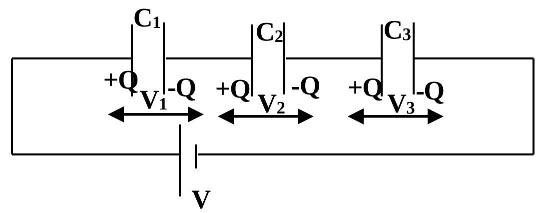
***(i) Parallel arrangement***




The diagram above shows three capacitors connected in parallel to the **same p.d, V.**
The charges on the individual capacitors are respectively,
$Q_1 = C_1V, \quad Q_2 = C_2V, \quad Q_3 = C_3V, \quad Q = CV$
Where, Q is the total charge of the system, and C is a single capacitor which is equivalent to the capacitance of the system.

Total charge of the system, Q is given by;
$$Q = Q_1 + Q_2 + Q_3$$
$$CV = C_1V + C_2V + C_3V$$
$$CV = V(C_1 + C_2 + C_3)$$
$$\boldsymbol{C = C_1 + C_2 + C_3}$$

***(ii) Series arrangement***

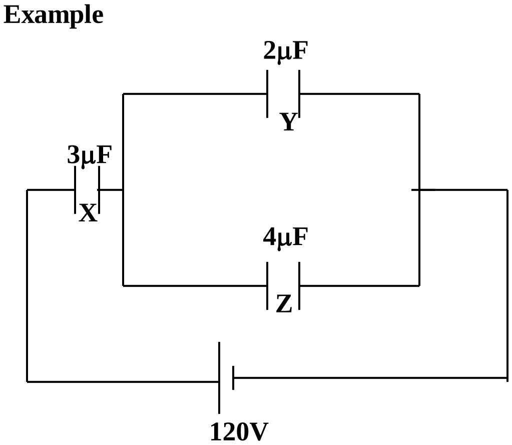



When a cell is connected in series, same charge appears on each capacitor plates. The p.d across the individual capacitors is therefore given by;
$$V_1 = \frac{Q}{C_1}, \quad V_2 = \frac{Q}{C_2}, \quad V_3 = \frac{Q}{C_3}, \quad V = \frac{Q}{C}$$
Where, Q is the total charge of the system, and C is a single capacitor which is equivalent to the capacitance of the system.
Total P.d, V of the system, is given by;
$$V = V_1 + V_2 + V_3$$
$$\frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}$$
$$\frac{Q}{C} = Q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)$$
$$\boldsymbol{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}$$

*Case of two capacitors in parallel*
$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \Leftrightarrow \frac{1}{C} = \frac{C_1 + C_2}{C_1C_2} \Leftrightarrow C = \frac{C_1C_2}{C_1 + C_2}$$

$$\mathbf{C = \frac{Product\ of\ capacitances}{Sum\ of\ capacitances}}$$

**Example**




Find (i) the p.d across each capacitor
(ii) Energy stored in the system of capacitors above.
(iii) Charge on Z.
(iv) Energy stored on capacitor Z

**Solution;**
(i) The p.d across each capacitor
Y and Z are in parallel, hence their effective capacitance,

$C_{YZ} = 2 + 4 = 6ìF$

X and YZ are in series, hence their effective capacitance,

$C_{XYZ} = \dfrac{6 \times 3}{6 + 3} = 2ìF$

Total charge flowing in the system.,

$Q = CV = 2 \times 10^{-6} \times 120 = 240 \times 10^{-6} = 2.4 \times 10^{-4} C$

Hence: $Q_Y = 2.4 \times 10^{-4} C$

Hence p.d across X,

$V_X = \dfrac{Q_X}{C_X} = \dfrac{2.4 \times 10^{-4}}{3 \times 10^{-6}} = 80V$

Hence P.d across YZ,

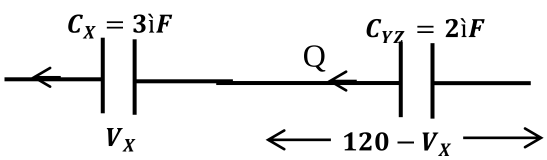
$V_{YZ} = 120 - 80 = 40V$

$V_Y = V_Z = 40V$

Since they are in parallel

(ii) Energy stored in the system of capacitors.

$W = \dfrac{1}{2} CV^2 = \dfrac{(2.0 \times 10^{-6})(120)^2}{2} = 0.0432 \text{ J}$




Therefore $Q_y = C_y V_y = 2 \times 10^{-6} \times 40 = 80 \times 10^{-6} C$

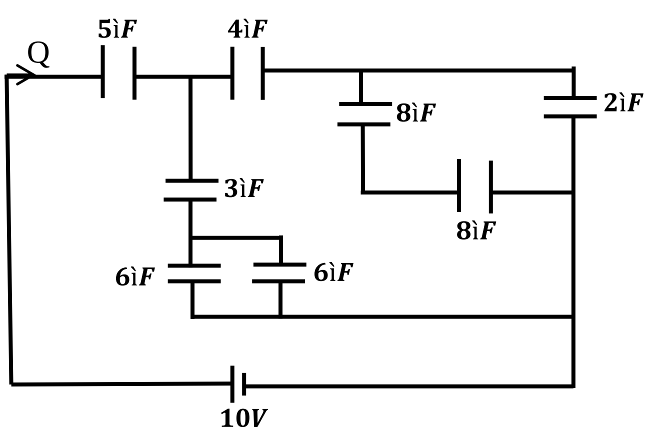
$Q_z = C_z V_z = 4 \times 10^{-6} \times 40 = 160 \times 10^{-6} C$

(iv) Energy stored on capacitor Z

$W = \dfrac{1}{2} C_Z {V_Z}^2 = \dfrac{(4.0 \times 10^{-6})(40)^2}{2} = 0.0032\text{J}$

**Example 2**

The figure below shows a net work of capacitors connected to a battery. Calculate the total energy stored in the net work.




Solution

***Simplify each branch separately, taking into account the polarity of each capacitor with respect to the direction of flow of charge.***

The 8μF capacitors are in series. Also the 8μF and the 2μF are in parallel.

- Effective capacitance of the 8μF capacitors in series is;

$\dfrac{1}{C_1} = \dfrac{1}{8} + \dfrac{1}{8} = \dfrac{1}{4} \Leftrightarrow C_1 = 4ìF$

$C_1$ and the 2μF capacitor are in parallel.

- Effective capacitance of $C_1$ and 2μF capacitors in parallel is;

$C_2 = 4 + 2 \Leftrightarrow C_2 = 6ìF$

The 6μF capacitors are in parallel.

- Effective capacitance of the 6μF capacitors in parallel is;

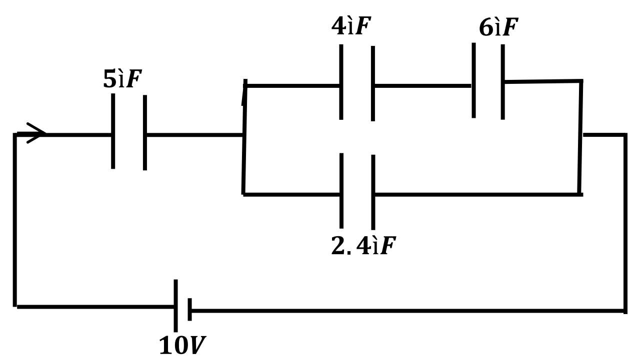
$C_3 = 6 + 6 \Leftrightarrow C_3 = 12ìF$

$C_3$ and the 3μF capacitor are in series.

- Effective capacitance of $C_3$ and the 3μF capacitors in series.

$\dfrac{1}{C_4} = \dfrac{1}{12} + \dfrac{1}{3} = \dfrac{5}{12} \Leftrightarrow C_4 = 2.4ìF$

Simplified (or amended) diagram.




The 4μF and 6μF capacitors are in series, thus;

$\dfrac{1}{C_s} = \dfrac{1}{4} + \dfrac{1}{6} = \dfrac{5}{12} \Leftrightarrow C_s = 2.4ìF$

$C_s$ and the 2.4μF capacitor are in parallel: Thus;

$C_p = 2.4 + 2.4 \Leftrightarrow C_p = 4.8ìF$

Now, $C_p$ and the 5μF capacitors are in series, thus; the coverall capacitance of the net work is;

$$\frac{1}{C_T} = \frac{1}{4.8} + \frac{1}{5} \Leftrightarrow C_T = 2.449ìF$$

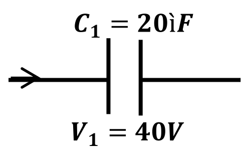
Therefore, the energy stored in the network is;

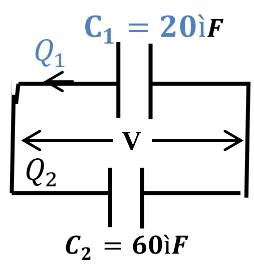
$W = \dfrac{1}{2} C_T {V_T}^2 = \dfrac{(2.449 \times 10^{-6})(10)^2}{2} = 1.22 \times 10^{-4}\text{J}$

**Example 2**

A 20μF capacitor is charged 40V and then connected across an un charged 60μF capacitor. Calculate the potential difference across the 60μF capacitor.

Solution




Charge Q initially on $C_1$ is given by;

$Q = C_1V_1$

$Q = (20 \times 10^{-6}) \times 40$

$\underline{Q = 8.0 \times 10^{-4}C}$

On connecting them in parallel, the charge Q is divided as per the ratio of their capacitances.

$$\frac{Q_1}{Q_2} = \frac{C_1}{C_2} \Leftrightarrow \frac{Q_1}{Q_2} = \frac{20}{60} = \frac{1}{3} \ ; i.e: Q_1 : Q_2 = 1:3$$

$$\text{Thus} Q_1 = \frac{1}{4} \times Q = \frac{1}{4} \times 8.0 \times 10^{-4} = 2.0 \times 10^{-4}C$$

From;

$Q_1 = C_1V$

$2.0 \times 10^{-4} = (20 \times 10^{-6})V$

$\underline{V = 10V}$

This is the same p.d across the 60μF capacitor since they are in parallel.

Alternatively;

From; $Q = Q_1 + Q_2$

$Q = C_1V + C_2V$

$Q = (C_1 + C_2)V$

$8.0 \times 10^{-4} = [(20 \times 10^{-6}) + (60 \times 10^{-6})]V$

$\underline{V = 10V}$

OR

$Q_2 = C_2V$

$2.0 \times 10^{-4} = (60 \times 10^{-6})V$

$\underline{V = 10V}$

**Example3:**

An air capacitor of capacitance 400μF is charged to 180V and connected across an un charged capacitor of capacitance 500μF.

(a) Find the energy stored in the 500μF capacitor.

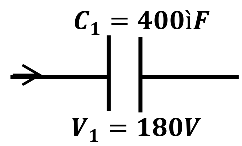
(b) With the two capacitors still connected together, a dielectric constant 1.5 is inserted between the plates of the 400μF. Find the new p.d across the two capacitors if ;

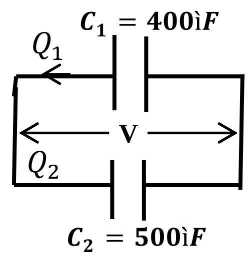
(i). The separation between the plates is remains constant.

(ii). The separation between the plates is doubled.

(iii). Explain the difference in the two p.ds in (b) (i) and (ii) above.

Solution




(i)

Charge Q initially on $C_1$ is given by;

$Q = C_1V_1$

$Q = (400 \times 10^{-6}) \times 180$

$\underline{Q = 7.2 \times 10^{-2}C}$

On connecting them in parallel, the charge Q is divided as per the ratio of their capacitances.

$$\frac{Q_1}{Q_2} = \frac{C_1}{C_2} \Leftrightarrow \frac{Q_1}{Q_2} = \frac{400}{500} = \frac{4}{5} \ ; i.e: Q_1 : Q_2 = 4:5$$

$$\text{Thus} Q_1 = \frac{4}{9} \times Q = \frac{4}{9} \times 7.2 \times 10^{-2} = 3.2 \times 10^{-2}C$$

$$\text{And} Q_2 = \frac{5}{9} \times Q = \frac{5}{9} \times 7.2 \times 10^{-2} = 4.0 \times 10^{-2}C$$

Energy stored in the 500μF capacitor;

$$W = \frac{Q_2^2}{2C_2} = \frac{(4.0 \times 10^{-2})^2}{2 \times 500 \times 10^{-6}} = 1.6\text{J}$$

(ii) When a dielectric is inserted btn the plates of $C_1$, the new capacitance $C_1'$is given by;

$$\frac{C_1'}{C_0} = å_r \Leftrightarrow C_1' = å_rC_0 = 1.5(400 \times 10^{-6}) = 600 \times 10^{-6}F$$

Thus;

1. $\frac{Q_1}{Q_2} = \frac{C_1}{C_2} \Leftrightarrow \frac{Q_1}{Q_2} = \frac{600}{500} = \frac{6}{5} \ ; i.e: Q_1 : Q_2 = 6:5$

$$Q_2 = \frac{5}{11} \times Q = \frac{5}{11} \times 7.2 \times 10^{-2} = 3.27 \times 10^{-2}C$$

From;

$Q_2 = C_2V$

$3.27 \times 10^{-2} = (500 \times 10^{-6})\text{V}$

$\underline{\text{V} = 65.4\text{V}}$

**Exercise**

1. The capacitance of a variable radio capacitor can be charged continuously from 10μF to 900μF by turning a diode from $0^0$ to$140^0$ with the diode set at $140^0$, the capacitor connected to a 9V battery. After charging, the capacitor is disconnected from the battery and the diode turned to $0^0$ . Calculate the;
   (a) Charge on the capacitor.[Q=0.0081C]
   (b) Energy stored on the capacitor with diode set at $140^0$ .[E = 0.03645J ]
   (c) Work done in turning the diode from $140^0$ to $0^0$ . Neglect the frictional forces.[E = 3.28J ]

2. **UNEB; 1998, 2008No. 10 (c). 2004(d), 2009No.10(c),**

# CURRENT ELECTRICITY

**Terms used;**

**Charge, Q;** Is the quantity of electricity that passes a given point in a conductor at a given time.

The S.I unit of charge is a coulomb. **A coulomb** is the quantity of electric charge that passes a given point in a conductor when a steady current of 1A flows in one second.

**Current, (I);** Is the rate of flow of charge. i.e. $\mathrm{I} = \frac{\mathrm{dQ}}{\mathrm{dt}}$.

The S.I unit of current is an **ampere**. An ampere is a current flowing in a circuit when a charge of one coulomb passes any point through the circuit in one second.

$1A = 1Cs^{-1}$

**Potential difference (P.d);** Is the work done in transferring one coulomb of charge from one point to another in a circuit.

Whenever current flows, it does so because the electric potential at two points are different. If the two points are at the same potential, no current flows between them. $\mathrm{P.d} = \frac{\mathrm{W}}{\mathrm{Q}}$

The S.I unit is a volt. A **volt** is the potential difference between two points in circuit in which, 1J of work is done in transferring 1C of charge from one point to another.

$1V = 1JC^{-1}$

**Electromotive force, (e.m.f):** Is the work done in transferring one coulomb of charge around a complete circuit in which a battery is connected.

It is the p.d across a cell in an open circuit.

<u>Sources of electrical e.m.f.</u>

(i). Electric cell: This converts chemical energy to electrical energy.

(ii). Generators: These convert mechanical energy to electrical energy.

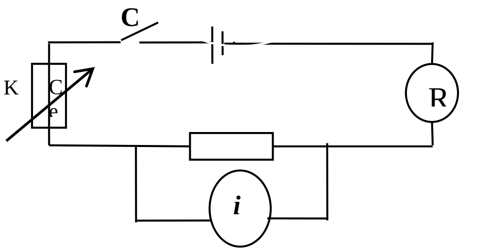
(iii). Thermo couple: This converts thermal energy (or heat energy) to electrical energy.

**Electrical Resistance, (R):** Is the opposition to the flow of current in a conductor. $\mathrm{R} = \frac{\mathrm{V}}{\mathrm{I}}$.

The S.I unit of resistance is an ohm (Ω). **An ohm** is the resistance of a conductor through which a current of one ampere flows when a p.d across it is one volt.

**Ohms law:**
It states that the current through an ohmic conductor is directly proportional to the P.d across it provided the physical conditions remain constant.

**Experiment to verify Ohms law;**




Switch, K is closed, and a current, I flows through the circuit and it is read from the ammeter and the corresponding voltmeter reading also read and recorded.
The rheostat is adjusted to obtain several values of V and I.
Plot a graph of V against I
It is a straight line graph through the origin, implying that V is directly proportional to I which verifies Ohm's law.
From the graph; .Slope, $\mathrm{R} = \frac{V}{\mathrm{I}} = \tan$ è.
Where è is the angle between the line and the horizontal.

**Factors affecting resistance of a conductor.**
The resistance of a conductor is independent of the P.d, V and the current I through the conductor but it depends on physical factors like; length, cross sectional area and temperature.

**(i). Length, l**
In conductors, electrons are loosely bound to the nucleus of the atoms.
Electrons make frequent collisions with the atoms as they move within the conductor.
When the length of the conductor is increased, electrons make more frequent collisions with the atoms. This reduces the drift velocity of the free electrons and hence increases the resistance of the conductor.

**(ii). Cross sectional area, A**
When there is an increase in the cross sectional area of the conductor, the number of free electrons that drift along the conductor also increases.
This means that there is an increase in the number of electrons passing a given point along the conductor per second, thus an increase in current
Consequently, this reduces the resistance of the conductor.

**(iii). Temperature, T**
When there is an increase in the temperature of the conductor, the atoms vibrate with greater amplitude and frequency about their mean positions.

The thermal velocity of the free electrons increases which increases their thermal energy. Consequently, the number of collisions between the free electrons and the atoms increases.
This leads to a decrease in the drift velocity of the electrons. This means that there is a decrease in the number of electrons passing a given point along the conductor per second, thus a decrease in current
Consequently, this increases the resistance of the conductor.
Note: **Supper conductors** are materials whose resistance vanishes when they are cooled to a temperature near$-273^0C$.

Combining the first two factors at constant temperature, we get:

$$R \propto \frac{l}{A} \Leftrightarrow \boldsymbol{R} = \text{ñ}\frac{\boldsymbol{l}}{\boldsymbol{A}}$$

Where, ñ is a constant which depends on the nature of the conductor. It is called the Resistivity of the conductor.

**Resistivity,** ñ**:**
**I**s the electrical resistance across the opposite faces of a cube of 1m length.
The S.I unit of resistivity is an ohm metre,(Ωm).

**Conductivity,** ó**:**
Is the reciprocal of electrical resistivity.

$$\text{ó} = \frac{\mathbf{1}}{\text{ñ}}$$

**Temperature coefficient of resistance:**
Is the fractional increase in the resistance of a material at $0^0C$ per one kelvin rise in temperature.
Suppose the resistance of a material at $0^0C = R_0$, and its resistance at è$^0C = R_{\text{è}}$,
Then the temperature change is$(\text{è} - 0)^0C = \text{è}^0C$.
Temperature coefficient of resistance

$$= \frac{\text{Change in resistance}}{\text{Resistance at } 0^0\text{C} \times \text{Temp. change.}}$$

$$\propto = \frac{\mathrm{R_\theta - R_0}}{\mathrm{R_0\theta}}$$

$$\mathbf{R}_{\text{è}} = \mathbf{R_0}(\mathbf{1} + \propto \text{è})$$

The S.I unit of temperature coefficient of resistance is$\boldsymbol{K^{-1}}$

**Metals have a positive temperature coefficient of resistance** because, their resistance increases with increase in in temperature.
Increase in temperature increases the kinetic energy of the electrons and hence their speed and amplitude of vibration of the atoms thereby leaving very little space for the free passage of electrons.

The rate of collision of the electrons with the vibrating atoms increases which reduces the drift velocity of the electrons. The rate of flow of charge or current decreases, which implies an increase in resistance of the metal substance.

**Semi- conductors have negative temperature coefficient of resistance** because their resistance decreases with increase in temperature.

Increase in temperature of a semi-conductor increases the thermal energy of the substance which enables it to overcome the strong nuclear forces of attraction (binding forces) and sets more electrons in the conduction band.

Increase in temperature also increases the kinetic energy of the free electrons and hence their speed leading to an increase in the drift velocities

The rate of flow of charge or current increases, which implies a decrease in resistance of the semi conductor.

**Example1**
The temperature coefficient of resistance of a wire is $0.5 \times 10^{-3} K^{-1}$.Find the length of a wire of diameter 2mm and resistivity$1.0 \times 10^{-6} \Omega m$. at $25^0 C$ needed to make a coil of resistance 6Ω at $95^0 C$.

**Solution:**
$\mathbf{R}_{è} = \mathbf{R_0}(\mathbf{1} + \propto è)$
$\mathbf{R_{25}} = \mathbf{R_0}(1 + 0.5 \times 10^{-3} \times 25)$……….. (i)
$\mathrm{R}_{95} = \mathrm{R}_0(1 + 0.5 \times 10^{-3} \times 95)$**……….. (i)**

**Equations (i) divide by Equation (ii)**

$$\frac{\mathbf{R_{25}}}{\mathbf{R_{95}}} = \frac{\mathrm{R}_0(1 + 0.5 \times 10^{-3} \times 25)}{\mathrm{R}_0(1 + 0.5 \times 10^{-3} \times 95)}$$

$$\frac{\mathbf{R_{25}}}{6} = \frac{1.0125}{1.0475}$$

$$\mathrm{R}_{25} = 5.8\Omega$$

$$R = ñ\frac{l}{A}\ ; where, A = \frac{ðd^2}{4} = \frac{ð \times (2 \times 10^{-3})^2}{4}$$
$$= 3.14 \times 10^{-6} m^2$$

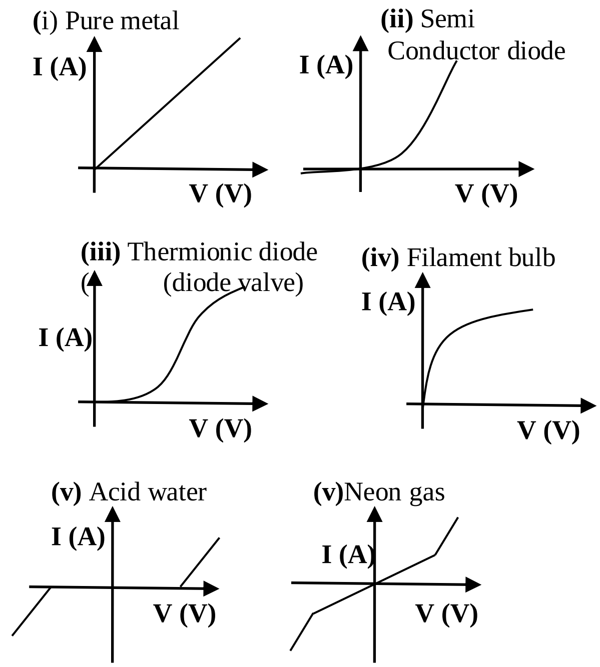
$$5.8 = 1.0 \times 10^{-6} \times \frac{l}{3.14 \times 10^{-6}}$$
$\underline{l = 18.22m}$

**Example;2**
A metal wire 1m long having a cross sectional area of $1.0 \times 10^{-6} m^2$ has a resistance of 0.20Ω at $0^0 C$. Calculate the resistivity of the metal at $500^0 C$ given that its temperature coefficient of resistance is $6.2 \times 10^{-3} K^{-1}$.

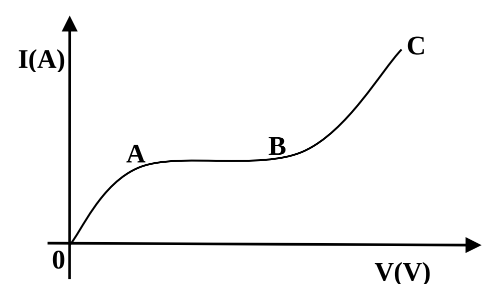
**Solution:**
$\mathbf{R}_{è} = \mathbf{R_0}(\mathbf{1} + \propto è)$
$\mathbf{R_{500}} = 0.2(1 + 6.2 \times 10^{-3} \times 500)$……….. **(i)**
$\mathbf{R_{500}} = 0.82\Omega$

$$R = ñ\frac{l}{A}$$
$$0.82 = ñ \times \frac{1}{1.0 \times 10^{-6}}$$
$ñ = 0.82(1.0 \times 10^{-6})$
$\underline{ñ = 8.2 \times 10^{-7} \Omega m}$

**Ohmic and non- ohmic conductors**
Ohmic conductors are conductors which obey Ohm's law.




<u>A sketch of current versus voltage for characteristic for a gas discharge tube</u>.




Along OA, the p.d is low so that the electrons and the positive ions do have low velocities.

Consequently, the current flowing is low. The current flowing is also proportional to the applied P.d so Ohm's law is obeyed.

Along AB, all the ions produced by the ionizing agent are collected by the electrode. So a constant current called saturation current flows.

Also in this region, the different velocity is sufficient to prevent appreciable recombination of positive ions and electrons.

Along BC, the p.d is high enough to enable the produced ions to have sufficient kinetic energy for knocking electrons out of the atoms they collide with. At the higher p.ds, even the electrons produced by collision acquire enough K.E to enable them to cause ionization when they collide with neutral gas atoms. So there is an un controllable growth of current and the gas is said to have broken down.

**Mechanism of Conduction of electricity in metals**
**(Heating effect of electricity)**

Metals have electrons that are loosely bound to the nuclei and are free to wander randomly through the metal lattice from one point to the other.

When the p.d is applied across the ends of the metal, an electric filed is set up which accelerates the electrons.

The accelerated electrons collide with atoms vibrating about a fixed mean position and give out some of their energy to the atoms.

The electrons are decelerated instantaneously, after which they are accelerated by the same field in the same direction hence they acquire a constant average drift velocity, in the direction from the negative to the positive terminals of the battery. It is this drift of electrons that constitute an electric current.

The amplitude of vibrations of atoms is however increased which increases the internal energy of the atoms as they rub against each other and the temperature of the atoms rises. Hence the heating action of an electric current (or the Joule heating).

In general, current flow in metals is due to movement of electrons (negative charge). In gases and electrolytes, both positive and negative charges are involved.

***Unit of charge: the coulomb (C)***

A coulomb is the quantity of electric charge carried past a given point in a circuit when a steady current of one ampere flows for one second.

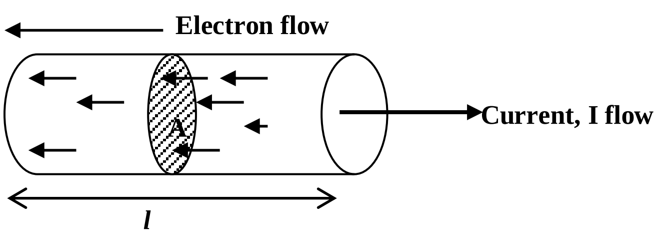
i.e. Charge, Q = current, I x time, t

Q = It

1C = 1As

**Drift velocity**

Consider a conductor of length ***l***, cross sectional area **A**, having **n** electrons per unit volume, each carrying a charge e.




Volume of the conductor = Al

Total number of electrons in the conductor = n(Al)

Total charge in the conductor = ne(Al)

If t is the time taken for an electron to drift through a distance l at a speed v is given by; $t = \frac{l}{v}$

From the definition of current; I

$$\mathbf{I} = \frac{Q}{t} = \frac{ne(Al)}{(l/v)} = \boldsymbol{neVA}$$

Thus the drift velocity, V is given by; $\mathbf{V} = \frac{\mathbf{I}}{\mathbf{neA}}$

**Current density (J);**

The current density at a point in a conductor is the current per unit cross sectional area at right angles to the direction of current flow at that point.

$$\mathbf{J} = \frac{I}{A} = \frac{neVA}{A} = \boldsymbol{neV}$$

It can also be defined as the amount of current flowing through a conductor of cross-sectional area 1m$^2$.

**Example**

1. A current of 10A flows through a copper wire of area 1mm$^2$. The number of free electrons per m$^3$ is $10^{29}$. Find the drift velocity of the electron.

$$v = \frac{I}{nAe} = \frac{10}{10^{29} \times 1.6 \times 10^{-19} \times 1 \times 10^{-6}} = 6.25 \times 10^{-4}\, ms$$

2. A metal wire contains $5x10^{22}$ electrons per cm$^3$ and has cross sectional area off 1mm$^2$. If the electrons move along the wire with a mean drift velocity of 1mms$^{-1}$, calculate:

(i) Current density

(ii) Current in the wire.

<u>Solution</u>

(iii) current density

$$\frac{I}{A} = nev = 5 \times 10^{22} \times 10^{6} \times 10^{-3} \times 1.6 \times 10^{-19} = 8 \times 10^{6}\, Am^{-2}$$

(iv) current = current density x Area, A =

$$\mathrm{I} = \sigma \mathrm{A} = (8 \times 10^{6}) \times 10^{-6} = 8\mathrm{A}$$

**Ohm's Law**

Under constant physical conditions, the potential difference across a conductor is proportional to the current through it.

$V \propto I$, hence $\frac{V}{I} = cons\tan t$

The constant is the resistance R of the conductor.

Hence $V = IR$

The unit of resistance is the ohm (Ω)

The ohm is the resistance of a conductor in which a current of one ampere flows when a p.d of one volt is applied across it.

**Example:**

A steady uniform current of 5mA flows along a metal cylinder of cross sectional area of 0.2mm$^2$, length, 5m and resistivity $3x10^{-5}$Ωm. find the p.d across the ends of the cylinder.

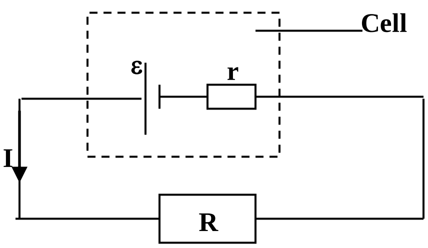
$$R = \rho \frac{l}{A} = \frac{3 \times 10^{-5} \times 5}{2 \times 10^{-7}} = 750\Omega$$

Hence $V = IR = 5\times10^{-3}\times750 = 3.75V$

***Question***

A p.d of 4.5V is applied to the ends of a 0.69m length of a wire of cross sectional area $6.6\text{x}10^{-7}\text{m}^2$. Calculate the drift velocity of electrons across the wire. (ρ of wire is $4.3\text{x}10^{-7}\Omega\text{m}$, number of electrons per $\text{m}^3$ is $10^{28}$ and electronic charge is $1.6\text{x}10^{-19}\text{C}$)

***Electromotive force, emf (ε) and internal resistance (r ) of the cell***




**Emf** of a source is the energy converted into electrical energy when 1C of charge passes through it.
Or it is the ratio of electrical power the source generates to the current which it delivers.

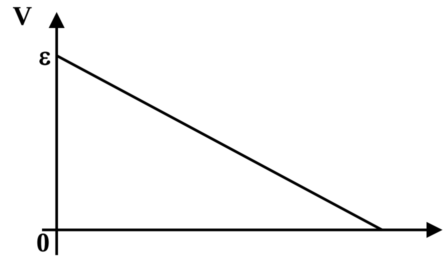
**Internal resistance** is the effective resistance of the source which accounts for the energy losses in it when it is supplying current. Internal resistance behaves as if it is a resistor in series with the battery.
It is the opposition to the flow of electric current in a cell due to its chemical composition.
Hence;
Energy supplied per coulomb by cell
= (energy changed per coulomb by an external circuit)
+ (energy wasted pre coulomb on internal resistance.)

$E.m.f = V + lost\ P.d$
Where V is the p.d across the external circuit. V is called terminal p.d and is the p.d across the cell in a closed circuit.
But lost p.d = Ir, where I is the current flowing through the circuit.
Hence;
$\text{å} = V + Ir$
$But\ also, V = IR$
$\text{å} = IR + Ir$
$\text{å} = \boldsymbol{I}(\boldsymbol{R}+\boldsymbol{r})$

$$\mathbf{I} = \frac{\mathbf{E.m.f}}{\mathbf{Total\ Resistance}}$$

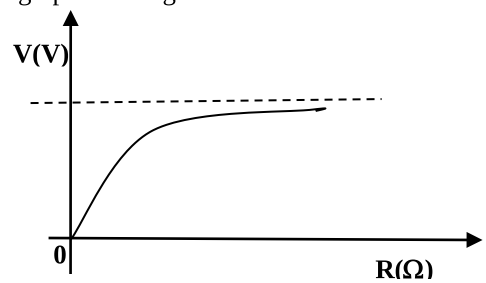
This is called the circuit formula.

A graph of V against I is shown below



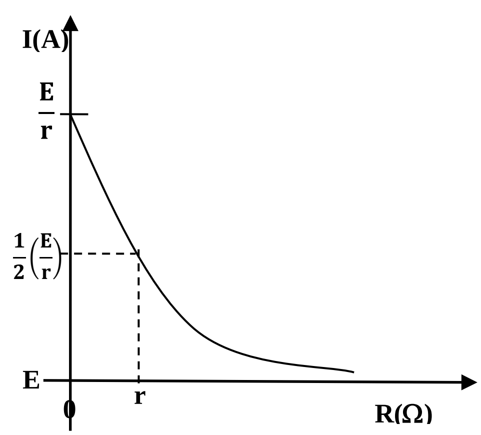

From the graph, it is seen that the e.m.f is equal to terminal p.d when current drawn from the cell is zero.
The negative slope of the graph is equal to the internal resistance of the cell.

A graph showing variation of P.d with resistance




From $\text{V} = \text{IR}$, and $\text{I} = \frac{\text{E}}{\text{R+r}}$
$\text{V} = \left(\frac{\text{E}}{\text{R}+\text{r}}\right)\text{R} = \frac{\text{ER}}{\text{R}+\text{r}}$
When R=0, V=0
As r increases, I decreases. since r is constant, it implies that the quantity Ir will decrease. But because E is constant, it follows that V= E- Ir will increase.
When R is very large, the terminal p.d approaches a constant value, E.

A graph showing variation of current with resistance.




From $\text{I} = \frac{\text{E}}{\text{R+r}}$, when R=0, $\text{I} = \frac{\text{E}}{\text{r}}$

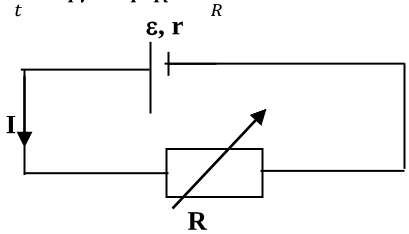
- When R is very small, the current approaches a constant value $\frac{\mathbf{E}}{\mathbf{r}}$**.** But the P.d (IR) falls steadily with R.
- When R= r, $\text{I} = \frac{1}{2}\left(\frac{\text{E}}{\text{r}}\right)$.

- When R I increased, the current I falls and the power (IV) falls with it.

**Electrical energy and power:**
If a charge Q is conveyed from one point to another in a circuit, in a time, t, such that a current I that flows is given by;
$Charge, Q = It \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(i)$
If a p.d between the is V, then;
$Work\ done, W = QV \ldots\ldots\ldots\ldots\ldots\ldots.(ii)$

From equations (i) and (ii)
$\mathbf{Work\ done, W = VIt}$
This work done is stored as electrical energy. Thus electrical energy is, E=VIt.

***Power out put and efficiency***
Electrical power is the energy liberated per second in a circuit.
$P = \frac{VIt}{t} = IV = I^2R = \frac{V^2}{R}$

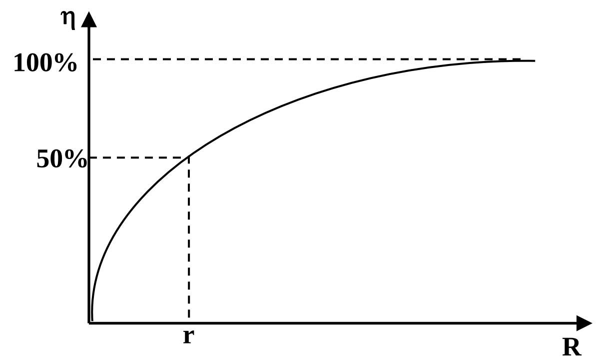



For a cell; $P = IE = I^2(R + r)$
From the circuit formula;
$$E = I(R + r) \Leftrightarrow I = \frac{E}{(R + r)}$$
$$V = IR$$
$$\Leftrightarrow V = \frac{ER}{(R + r)}$$

Efficiency,
$$ç = \frac{Power\ output}{Power\ in\ put} \times 100\%$$
$$ç = \frac{IV}{IE} \times 100\% = \frac{V}{E} \times 100\%$$
$$ç = \frac{V}{E} \times 100\% = \frac{\frac{ER}{(R+r)}}{E} \times 100\%$$
$$ç = \frac{\mathbf{R}}{(\mathbf{R} + \mathbf{r})} \times \mathbf{100}\% = \frac{\mathbf{1}}{\left(\mathbf{1} + \frac{r}{R}\right)} \times \mathbf{100}\%$$

If R = r, η = 50%; As R ⟶ ∞ , η ⟶ 100%

Hence a graph of η against Load resistance R is shown below:

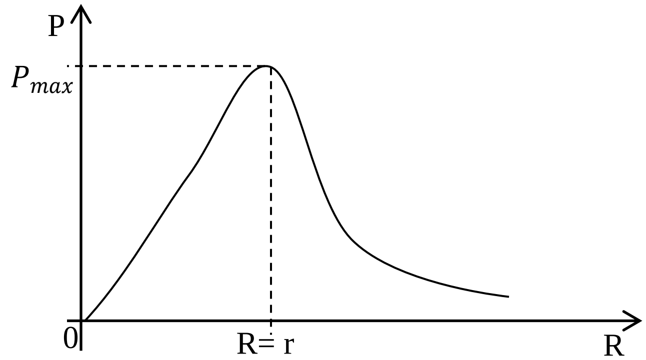



<u>Variation of power with Resistance.</u>
$$\text{Powe}, P = IV = I^2R, \text{Where } I = \frac{E}{(R + r)}$$
$$\text{Powe}, P = \left(\frac{E}{R + r}\right)^2 R = \frac{E^2R}{(R + r)^2}$$

For maximum power, $P_{max}$: $\frac{dP}{dR} = 0$
$$\frac{dP}{dR} = \frac{d}{dR}\left\{\frac{E^2R}{(R + r)^2}\right\} = 0$$
Using the quotient rule;
$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{dP}{dR} = \frac{(R + r)^2\frac{d(E^2R)}{dR} - E^2R\frac{d(R+r)^2}{dx}}{[(R + r)^2]^2}$$
$$\frac{dP}{dR} = \frac{E^2[(R + r)^2 - 2R(R + r)]}{(R + r)^4}$$
$$0 = \frac{E^2[(R + r)^2 - 2R(R + r)]}{(R + r)^4}$$
$\mathbf{R = r}$
$$\text{Power output}, P = \frac{E^2R}{(R + r)^2}$$
$$\text{Max. Power}, P = \frac{E^2R}{(R + R)^2} = \frac{E^2R}{(2R)^2} = \frac{E^2R}{4R^2} = \frac{E^2}{4R}$$
$$\mathbf{P_{max}} = \frac{\mathbf{E^2}}{\mathbf{4R}} = \frac{\mathbf{E^2}}{\mathbf{4r}}$$

As R tends to zero, P tends to zero
As R tends to ∞, P tends to zero.
<u>A graph of power out put P against load resistance R is shown below.</u>

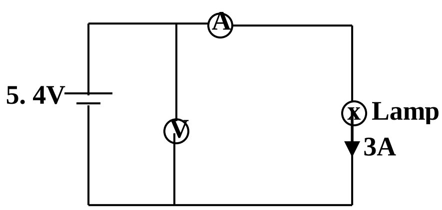
This power is generated as in for of heat energy perunit time.




Exercise

1. When a 10Ω resistor is connected across the terminals of the cell of emf, E and internal resistance, r, a current of 0.1A flows through the resistor. If the 10Ω is replaced with a 3Ω resistor, the current increases to 0.24A. Find E and r.

2.



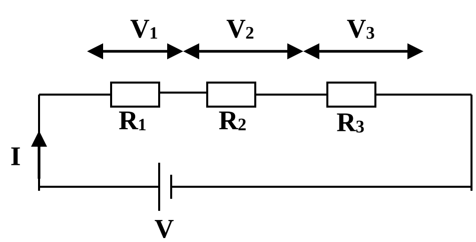

In the circuit above, V reads 4.8V. Calculate

(i) the internal resistance of the cell
(ii) the energy transformed per second in the lamp
(iii) State any two assumptions made in calculations.

3. A voltmeter with resistance 20KΩ is connected across the power supply and gives a reading of 44V. Another voltmeter with a resistance of 50KΩ connected across the same supply gives a reading of 50V. Find the emf of the power supply.

**Resistor Net works**

***(i) Series arrangement of resistors***




In series

(i) same current flows through each resistor

(ii) total p.d V = sum of p.d across each resistor.

Hence $V = V_1 + V_2 + V_3$

Using ohm's law, $V_1 = IR_1$, $V_2 = IR_2$ and $V_3 = IR_3$

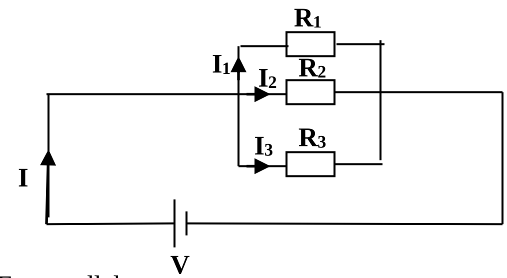
Hence $V = I(R_1 + R_2 + R_3)$

If R is the resistance of a single resistor representing the three resistors then V=IR

Hence $IR = I(R_1 + R_2 + R_3)$

$$R = (R_1 + R_2 + R_3)$$

***(ii) Parallel arrangement of resistors***




For parallel

(i) same p.d across each resistor
(ii) Total current, I is equal to sum of Current through each resistor.

Hence $I = I_1 + I_2 + I_3$

Using ohm's law, $V = I_1R_1$, $V = I_2R_2$ and $V = I_3R_3$

$$I = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3}$$

$$I = V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

If R is the resistance of a single resistor representing the three resistors then,

V =IR

Hence $\frac{1}{R} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$

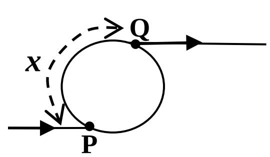
***Question***

Show that for two resistors in parallel, the effective resistance

$$R = \frac{R_1R_2}{R_1 + R_2}$$

**Example 1**

1. A wire of diameter ***d,*** length ***l*** and resistivity ***ρ*** forms a circular loop. Current enters and leaves at points **P** and **Q** respectively**.**




Show that the resistance R of the wire is given by ;

$$R = \frac{4ñx(l - x)}{ðd^2 l}$$

Let $R_1$ and $R_2$ be resistance of portion $x$ and $(l-x)$ of the wire respectively.

$$R_1 = \frac{ñx}{A}, \quad R_2 = \frac{ñ(l-x)}{A}, and A = \frac{ðd^2}{4} \ldots\ldots\ldots (i)$$

The two portions are in parallel, hence;

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Leftrightarrow R = \frac{R_1 R_2}{R_1 + R_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Put (i) into (ii);

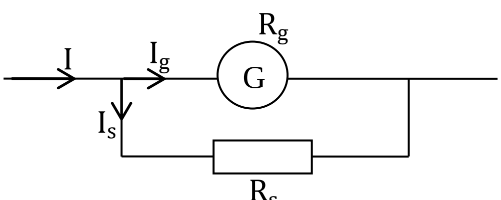
$$R = \frac{R_1 R_2}{R_1 + R_2} = \frac{\left(\frac{ñx}{A}\right)\left[\frac{ñ(l-x)}{A}\right]}{\frac{ñx}{A} + \frac{ñ(l-x)}{A}} = \frac{ñx(l-x)}{Al}$$

$$\Leftrightarrow R = \frac{4ñx(l-x)}{ðd^2 l}$$

**CONVERSION OF A MOVING COIL GALVANOMETER INTO AMMETERS AND VOLTIMETERS.**

Use of Shunts and Multipliers

**(i). Use of shunts**

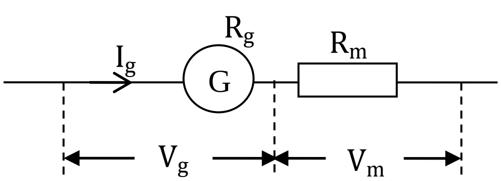



$$I_s = I - I_g$$

$I_g$ is the full-scale deflection of the galvanometer

$$\text{P.d across the shunt} = \text{P.d across galvanometer}$$
$$\Leftrightarrow \mathbf{(I - I_g)R_s = I_g R_g}$$

**(ii). Use of multipliers**




$$V = V_m$$

$$V = (\text{P.d across the multiplier}) + (\text{P.d across galvanometer})$$

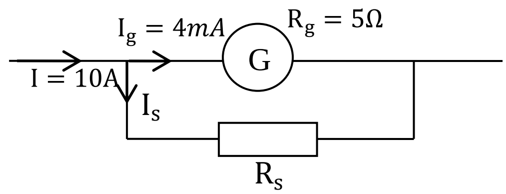
$$V = V_m + V_g$$
$$V = I_g R_m + I_g R_g$$
$$\mathbf{V = I_g(R_m + R_g)}$$

**Example1.**

A moving coil galvanometer gives a full scale deflection of 4mA and has a resistance of 5Ω.How can such instrument be converted into an ammeter giving a ful-scale deflection of 10A?

Solution:

Let Rs be the resistance of the shunt required.




$$\text{P.d across the shunt} = \text{P.d across galvanometer}$$
$$\Leftrightarrow (I - I_g)R_s = I_g R_g$$
$$\Leftrightarrow (10 - 0.004)R_s = 0.004 \times 5$$
$$\mathbf{R_s = 0.002Ù}$$

Thus a low resistance resistor of $0.002\Omega$ should be connected in parallel with the instrument.

**Examp2.**

A moving coil galvanometer gives a full scale deflection of 6mA and has a resistance of 4Ω.How can such instrument be converted into;

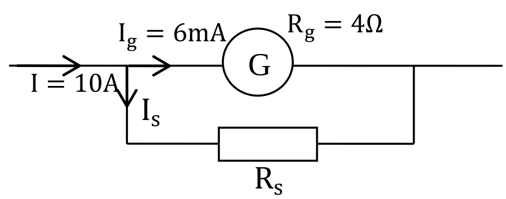
(i). an ammeter giving a ful-scale deflection of 15A?

(ii). A voltmeter reading up to 20V?

Solution:

(i)

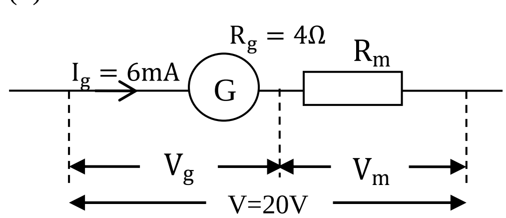
Let Rs be the resistance of the shunt required.




$$\text{P.d across the shunt} = \text{P.d across galvanometer}$$
$$\Leftrightarrow (I - I_g)R_s = I_g R_g$$
$$\Leftrightarrow (15 - 0.006)R_s = 0.006 \times 4$$
$$\mathbf{R_s = 0.0016Ù}$$

Thus a low resistance resistor of $0.0016\Omega$ should be connected in parallel with the instrument.

(ii)




$$V = (\text{P.d across the multiplier}) + (\text{P.d across galvanometer})$$
$$V = V_m + V_g$$
$$V = I_g R_m + I_g R_g$$

$$R_m = \frac{V - I_gR_g}{I_g}$$

$$R_m = \frac{20 - 0.006(4)}{0.006}$$

$\mathbf{R_m = 3329}$Ù

**Example 3.**

A moving coil galvanometer of resistance5Ω and current sensitivity of 2 divisions per milliampere, gives a full-scale deflection of 16 divisions. Explain how such an instrument can be converted into;

(i). An ammeter reading up to 20A?

(ii). A voltmeter in which each division represents 2V?

**Solution:**

(i)

Current sensitivity = 2div/mA

Full scale deflection = 16div.

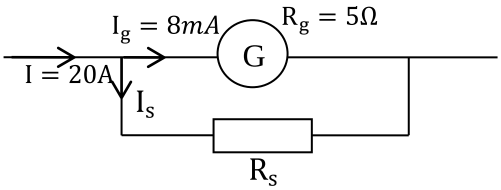
$2div \rightarrow 1mA$

$16div \rightarrow x$

On cross multiplying, we get;

$2x = 16mA$

$x = 8mA = 8 \times 10^{-3}A = 0.008A$




P. d across the shunt = P. d across galvanometer

$$\Leftrightarrow (I - I_g)R_s = I_gR_g$$

$$\Leftrightarrow (20 - 0.008)R_s = 0.008 \times 5$$

$\mathbf{R_s = 0.002}$Ù

Thus a low resistance resistor of $0.0016\Omega$ should be connected in parallel with the instrument.

Voltimeter sensitivity = 1div/2V

Full scale deflection = 16div.

$1div \rightarrow 2V$

$16div \rightarrow y$

On cross multiplying, we get;

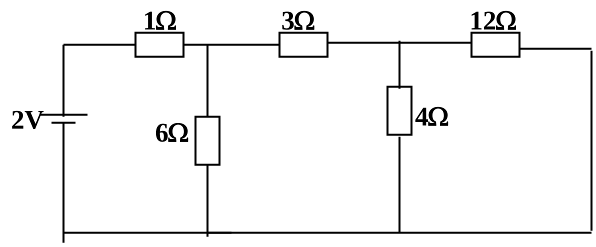
$y = 16 \times 2 = 32V$

$$R_m = \frac{V - I_gR_g}{I_g}$$

$$R_m = \frac{32 - 0.008(5)}{0.008}$$

$\mathbf{R_m = 3995}$Ù

**Example:4**




4 Ωand 12Ω resistors are parallel, their effective resistance is

$$R_1 = \frac{4 \times 12}{4 + 12} = 3\Omega$$

$R_1$ and 3Ω resistors are in series, their effective resistance is $R_2 = R_1 + 3 = 3 + 3 = 6\Omega$

$R_2$ and 6Ω resistors are in parallel, their effective resistance is

$$R_3 = \frac{6 \times 6}{6 + 6} = 3\Omega$$

$R_3$ and 1Ω resistors are in series, their effective resistance is

$$R = R_3 + 1 = 3 + 1 = 4\Omega$$

Hence effective resistance of the whole circuit is R = 4Ω

Current flowing $I = \frac{V}{R} = \frac{2}{4} = 0.5A$

**Example:5**

A battery of un known e.m.f and internal resistance is connected in series with a load of resistance, R ohms. If a very high resistance voltmeter is connected across the load reads 3.2V and the power is dissipated in the battery is 0.032W and efficiency of the circuit is 80%. Find the:

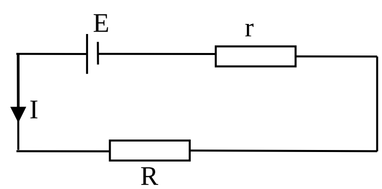
(i) Current flowing

(ii) Internal resistance of the battery.

(iii) Load resistance, R

(iv) E.m.f of the battery.

**Solution:**




Total resistance, = R + r

From the circuit formula;

$$I = \frac{E}{R + r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (i)$$

Power dissipated in the battery;

$P = I^2r = 0.032 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (ii)

From Ohm's law; the terminal p.d is;

$V = IR = 3.2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . \ldots$ (iii)

$$\text{Efficiency}, = \frac{\text{Power output}}{\text{Power input}} \times 100$$

$$\frac{80}{100} = \frac{I^2R}{ER}$$
$$0.8 = \frac{I^2R}{I^2(R+r)}$$
$$R = 0.8(R+r)$$
$$R = 4r$$

From equation (iii)
$I(4r) = 3.2 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (iv)$

Equation (ii) ÷ (iv)
$$\frac{I^2r}{4Ir} = \frac{0.032}{3.2}$$
$\underline{I = 0.04\ A}$

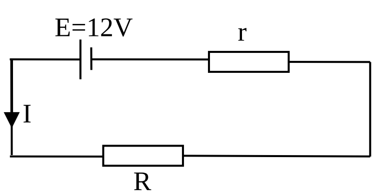
From equation (ii)
$(0.04)^2 r = 0.032$
$\underline{r = 20\Omega}$

$R = 4r$
$\underline{R = 4(20) = 80\Omega}$

From equation (ii)
$E = I(R+r)$
$E = 0.04(40+20)$
$\underline{E = 4\ V}$

**Example:6**
A battery of e.m.f 12V and un known internal resistance is connected in series with a load resistance, R reads 11.4V and the power dissipated in the battery is 0.653W. Find the:
(i) Current flowing.
(ii) Internal resistance of the battery.
(iii) Load resistance, R.
(v) Efficiency of the circuit.

**Solution:**




From Ohm's law; the terminal p.d is;
$V = IR = 11.4 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (i)$

P.d across the internal resistance is lost volts (Ir). From; $E = I(R+r) \Leftrightarrow Ir = E - IR$
$Ir = 12 - 11.4$
$Ir = 0.6 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (ii)$

Power dissipated in the battery;
$P = I^2r = 0.653 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (iii)$

Equation (iii) ÷ (ii)
$$\frac{I^2r}{Ir} = \frac{0.653}{0.6}$$
$\underline{I = 1.088\ A}$

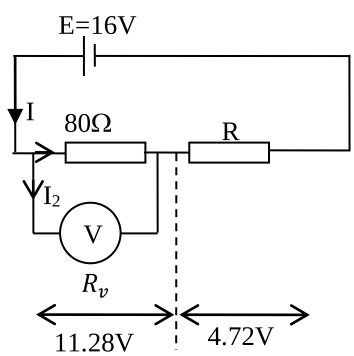
From equation (iii)
$(1.088)^2 r = 0.653$
$\underline{r = 0.55\Omega}$

From equation (i)
$IR = 11.4$
$(1.088)R = 11.4$
$\underline{R = 10.48\Omega}$

$$\text{Efficiency,} = \frac{\text{Power output}}{\text{Power input}} \times 100\%$$
$$= \frac{I^2R}{ER} \times 100\%$$
$$= \frac{(1.088)^2(10.48)}{12(1.088)} \times 100\%$$
$$= 95\%$$

**Example:7**
A battery of e.m.f 16V and negligible internal resistance is connected in series with a load resistance of 80Ω and RΩ. When a voltmeter is connected across the 80Ω resistor, it reads 11.28V while it reads 2.83V when connected across R. Find the:
(i) Value of R.
(ii) Resistance of the voltmeter.

**Solution:**
**Case I**

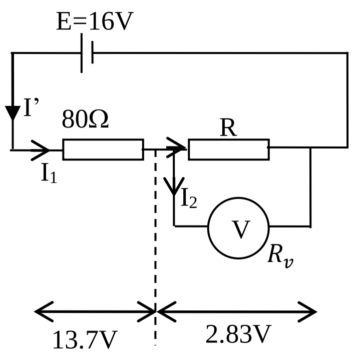



Let $R_1$ be the effective resistance of the 80Ω and the $R_v$.
$$\frac{1}{R_1} = \frac{1}{80} + \frac{1}{R_v} \Leftrightarrow \frac{1}{R_1} = \frac{80+R_v}{80R_v}$$
$$R_1 = \frac{80R_v}{80+R_v} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (i)$$

$I\ R_1 = 11.28$
$$I\left(\frac{80+R_v}{80R_v}\right) = 11.28 \dots\dots\dots\dots\dots\dots\dots\dots (ii)$$

$IR = 4.72 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (iii)$

**Case II**




Let $R_2$ be the effective resistance of the R and the $R_v$.

$$\frac{1}{R_2} = \frac{1}{R} + \frac{1}{R_v} \Leftrightarrow \frac{1}{R_2} = \frac{R + R_v}{R R_v}$$

$$R_2 = \frac{R R_v}{R + R_v} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

$$I' R_2 = 2.83$$

$$I'\left(\frac{R R_v}{R + R_v}\right) = 2.83 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(5)$$

$$I'(80) = 13.7 \ldots\ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(6)$$

$$I' = 0.165 \text{ A}$$

Thus;

$$0.165\left(\frac{R R_v}{R + R_v}\right) = 11.28$$

$$\frac{R R_v}{R + R_v} = 17.152$$

$$R R_v = 17.152R + 17.152R_v \ldots\ldots\ldots\ldots\ldots\ldots.(7)$$

Equation (iii) ÷ (4)

$$\frac{80R_v}{R(80 + R_v)} = \frac{11.28}{4.72}$$

$$\frac{80R_v}{R(80 + R_v)} = 2.39$$

$$80R_v = 2.39R(80 + R_v)$$

$$80R_v = 191.186R + 2.39RR_v \ldots\ldots.\ldots\ldots(8)$$

Put eqn.(7) into (8).

$$80R_v = 191.186R + 2.39(17.152R + 17.152R_v)$$

$$80R_v = 232.179R + 40.993R_v$$

$$R_v = 5.952R$$

Put eqn.( $R_v$) into (7).

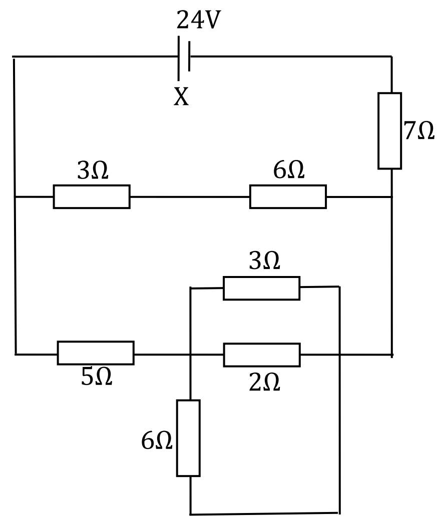
$$5.952R^2 = 17.152R + 102.089R$$

$$\underline{R = 20.03\Omega}$$

$$R_v = 5.952(20.03)\ \Omega$$

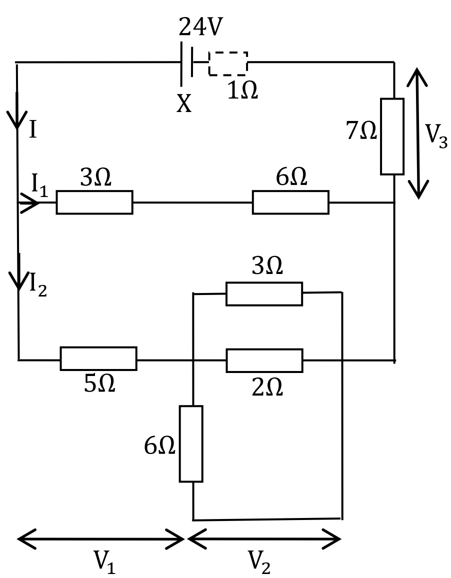
$$\underline{R_v = 119.04\ \Omega}$$

5. In the figure above, X is an accumulator of e.m.f 24V and internal resistance of 1Ω.




Find the effective resistance of the circuit and the current flowing in the circuit.

**Solution**

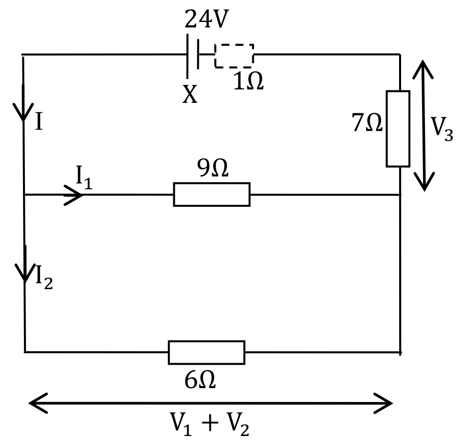



The 3Ω, 2Ω and 6Ω resistors are in parallel.

$$\frac{1}{R} = \frac{1}{3} + \frac{1}{2} + \frac{1}{6} = \frac{6}{6} = 1 \Leftrightarrow R = 1\Omega$$

The 3Ω and 6Ω are in series.
$R = 3 + 6 = 9\Omega$.

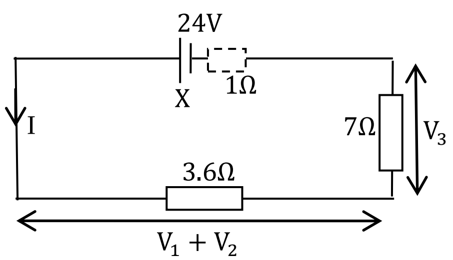
The 5Ω and 1Ω are in series.
$R = 5 + 1 = 6\Omega$.




The 9Ω and 6Ω resistors are in parallel.

$$\frac{1}{R} = \frac{1}{9} + \frac{1}{6} = \frac{5}{18} = 1 \Leftrightarrow R = 3.6\Omega$$

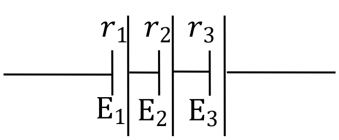
Equivalent circuit;




$\underline{\underline{\text{Total Resistance, } R_{Tot} = 3.6 + 7 + 1 = 11.6\Omega}}$

From Ohms law;
$V = IR$
$24 = 11.6I$
$\underline{\underline{I = 2.069\text{ A}}}$

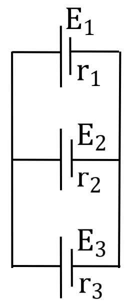
**Cells connected in series.**




The total e.m.f. E is given by;
$\mathbf{E = E_1 + E_2 + E_3}$
If any cell is connected in opposition to the direction of current, its e.m.f is assigned a negative sign.

The total internal resistance, r is given by;
$\mathbf{r = r_1 + r_2 + r_3}$

**Cells connected in series.**




When the cells are connected in parallel, the total e.m.f in the combination is the same as that in a single cell. i.e:
$\mathbf{E = E_1 = E_2 = E_3}$

The total internal resistance, r is given by;

$$\mathbf{\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}}$$

**Example:1**
A battery of e.m.f 18V and internal resistance of 3Ω is connected to a resistor of resistance 8Ω. Calculate the:
(i). Power generated[29.45W]
(ii). Efficiency[72.7%]

**Example:2**
Two cells each having an e.m.f of 1.5V and internal resistance 1Ω are connected to resistance of 4Ω. Calculate the current in this resistance if the cells are in:
(i). Parallel [0.33A]
(ii). Series [0.5A]

**Example:3**
A battery of e.m.f 2V and internal resistance of 1Ω is connected in series to a battery of e.m.f 4V and internal resistance of 3Ω so that they assist each. A resistor of 8Ω is joined to this arrangement. Calculate the current flowing:[I = 0.5A].

**Example:4**
A cell is joined in series with a resistor of 2Ω and a current of 0.25A flows through it. When a second resistor of 2Ω is connected in parallel with the first, the current through the cell increases to 0.3A. Calculate the e.m.f and internal resistance of the cell.

**Solution:**
**Case I**
I=0.25A, R=2Ω, E=?, r=?
$E = I(R + r)$
$E = 0.25(2 + r)$
$E = 0.5 + 0.25r \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$

**Case II**
I= 0.3A, $R_2 = \frac{RR}{R+R} = \frac{R^2}{2R} = \frac{2^2}{2(2)} = 1\Omega$, E=?, r=?

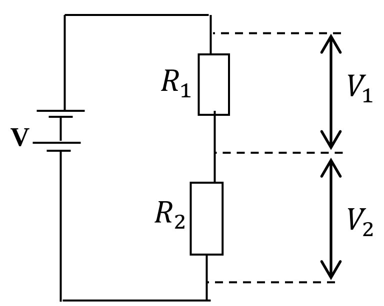
$E = I(R_2 + r)$
$E = 0.3(1 + r)$
$E = 0.3 + 0.3r \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(ii)$

Solving equations (i) and (ii) simultaneously, we get;
$\underline{E = 1.5V, \text{and } r = 4\Omega}$

**Example:6**
A d.c source of e.m.f 16V and negligible internal resistance is connected in series with two resistors of 400Ω and RΩ respectively. When the voltmeter is connected across the 400Ω resistor, it reads 4.0V, while it reads 6.0V when connected across the RΩ resistor. Find the;
(i). Resistance of the voltmeter [$R_v = 400\Omega$].
(ii). Value of R [$R = 600\Omega$].

**POTENTIAL DIVIDERS**
When resistors are arranged in series, they form what is called a potential divider.

Consider two resistors $R_1$ and $R_2$ connected in series with a d.c supply of V volts and negligible internal resistance.




Total resistance: $R = R_1 + R_2 \ldots\ldots\ldots$ (i)

From the circuit formula;
Current flowing, $I = \frac{\text{E.m.f}}{\text{Total resistance}}$
$I = \frac{V}{R_1+R_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (ii)

From Ohm's law;
$V_1 = IR_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (iii)
$V_2 = IR_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (iv)

Put Eqn.(ii) into Eqn.(iii) and Eqn.(iv)
$V_1 = \left(\frac{V}{R_1+R_2}\right)R_1.$

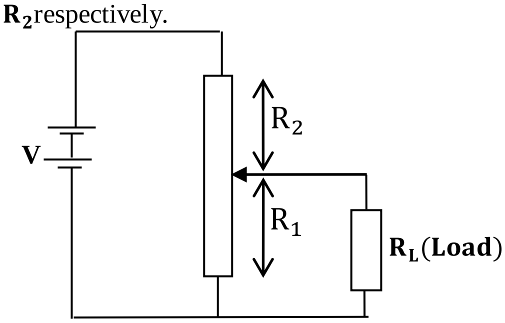
$V_1 = \left(\frac{R_1}{R_1+R_2}\right)V \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (v)

$V_2 = \left(\frac{R_2}{R_1+R_2}\right)V \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (vi)

Thus $\mathbf{V_1 : V_2 = R_1 : R_2}$

Therefore, for any resistors arranged in series, the p.d $V_i$ across $R_i$ is given by:
$$\mathbf{V_i = \left(\frac{R_i}{TotalResistance}\right) \times Supplyvoltage}$$

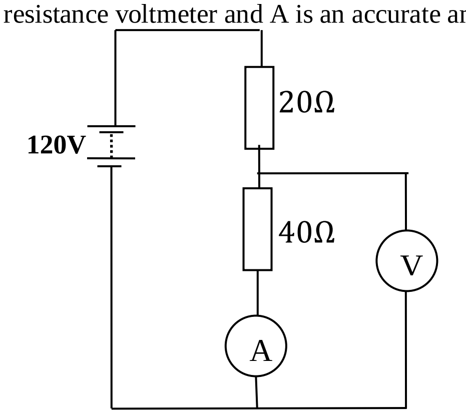
**Note:**
In potential dividers, the load is connected across a section of the divider whose P.d is equal to the operating voltage of the load. The load may be a cooker, an electric bulb, Radio or Tv sets, e.t.c. The load resistance, $\mathbf{R_L}$, will now be considered to be in parallel with the resistance of the lower section of the divider.
Let the resistance of the lower and upper sections be $\mathbf{R_1}$and $\mathbf{R_2}$respectively.




Now $R_1$ is in parallel with and $R_L$.
P.d across $R_1$= P.d across $R_L$.
Let $R_{eff}$ be the effective resistance of $R_1$ and $R_L$.
$$\frac{1}{R_{eff}} = \frac{1}{R_1} + \frac{1}{R_L}$$

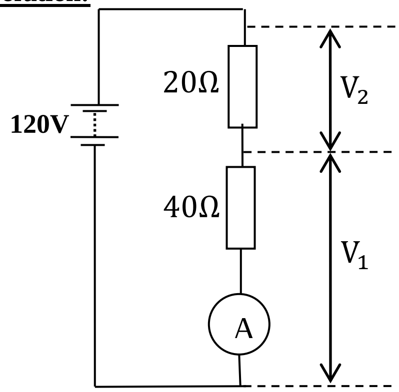
Now $R_{eff}$ is in series with $R_2$. Thus;
$\mathbf{Total\ resisrance\ ; R = R_{eff} + R_2}$

**Example: 1**
The figure below shows a potential divider. V is a very high resistance voltmeter and A is an accurate ammeter.




(i) Find the ammeter and voltmeter readings.
(ii) If the voltmeter was replaced by another voltmeter of resistance120Ω., what will be its new reading?
(iii) Find the percentage change in the ammeter reading.
(iv) If the voltmeter above is replaced by a C.R.O, what will be its new reading?

**Solution:**

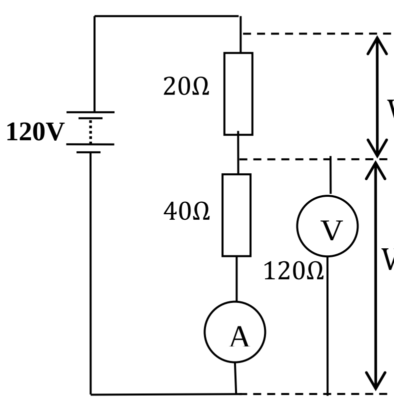



(i)
Total resistance; $R = 20 + 40 = 60\Omega$
From the circuit formula;
$$\mathrm{I} = \frac{\mathrm{E.m.f}}{\mathrm{Total\ resistance}}$$

$$\mathrm{I} = \frac{120}{60} = 2A$$
Thus the ammeter reading was 2 amperes.

(ii)
From Ohm's law;
$\mathrm{V} = \mathrm{IR}$
$\mathrm{V} = 2(40)$
$\mathrm{V} = 80\mathrm{V}.$

(iii)




Now the $40\Omega$ and $120\Omega$ are in parallel.
Let R be the effective resistance in parallel.
$$\frac{1}{\mathrm{R}} = \frac{1}{40} + \frac{1}{120}$$
$\mathrm{R} = 30\Omega$

Now R and the $20\Omega$ are in series. Thus;
Total resistance; $R = 20 + 30 = 50\Omega$
From the circuit formula;
$$\mathrm{I} = \frac{\mathrm{E.m.f}}{\mathrm{Total\ resistance}}$$

$$\mathrm{I} = \frac{120}{50} = 2.4A$$

Thus the ammeter reading will be 2.4 amperes.

From Ohm's law;
$\mathrm{V} = \mathrm{I}R_1$
$\mathrm{V} = 2.4(30)$
$\mathrm{V} = 72\mathrm{V}.$
Thus the new voltmeter reading will be 72Volts.

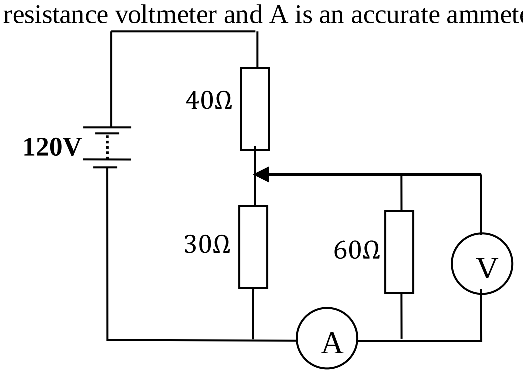
From Ohm's law;
$\mathrm{V} = I_1 R$
$120 = I_1(40)$
$I_1 = 1.8\mathrm{A}.$

*Percentage change in current will be;*

$$\%\ change; = \frac{Change\ in\ current}{Original\ value} \times 100\%$$
$$\%\ change; = \left(\frac{I - I_1}{I}\right) \times 100\%$$
$$\%\ change; = \left(\frac{2 - 1.8}{2}\right) \times 100\%$$
$\%\ change; = 10\%$

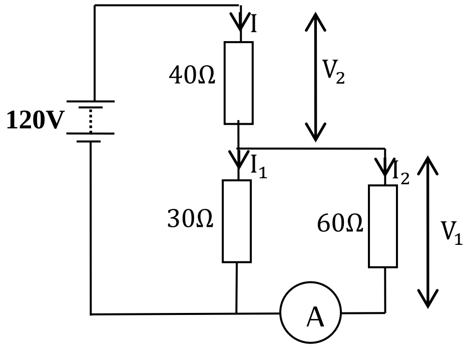
(v) A C.R.O is considered as an ideal voltmeter of infinite resistance, which makes it very accurate. Hence its reading would be the same as in (i) above. i.e. **80V**.

**Example:2**
The figure below shows a potential divider. V is a very high resistance voltmeter and A is an accurate ammeter.




(vi) Find the ammeter and voltmeter readings.
(vii) If the voltmeter was replaced by another voltmeter of resistance120Ω., what will be its new reading?
(viii) Find the percentage change in the ammeter reading.

**Solution.**
(i)




The 30Ω and 60Ω are in parallel.
Let R be the effective resistance in parallel.

$\frac{1}{R} = \frac{1}{30} + \frac{1}{60}$

$\underline{R = 20\Omega}$

Total resistance; $R = 20 + 40 = 60\Omega$

From the circuit formula;

$I = \frac{\text{E.m.f}}{\text{Total resistance}}$

$I = \frac{120}{60} = 2A$

Thus the ammeter reading was 2 amperes.

From Ohm's law;

$V = IR$

$V = 2(20)$

$\underline{V = 40V.}$

Thus the voltmeter reading was 40 volts.

From Ohm's law;

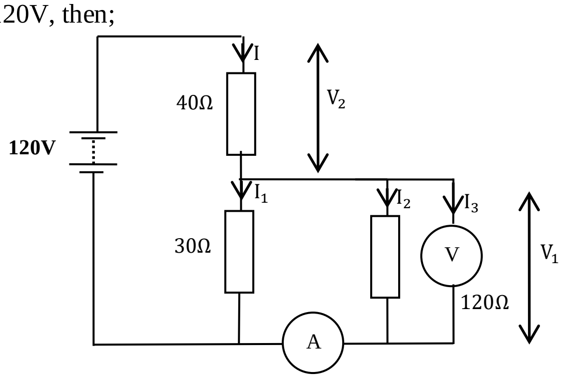
$V = I_2 R$

$40 = I_2(60)$

$I_2 = 0.667A.$

$I_1 = (I - I_2) = (2 - 0.667) = 1.333A$

(ii)
If the voltmeter is replaced by another voltmeter of resistance 120V, then;




The 30Ω, 60Ω and the 120Ω are in parallel.
Let R be the effective resistance in parallel.

$\frac{1}{R} = \frac{1}{30} + \frac{1}{60} + \frac{1}{120}$

$\underline{R = 17.143\Omega}$

Total resistance; $R = 17.143 + 40 = 57.143\Omega$

From the circuit formula;

$I = \frac{\text{E.m.f}}{\text{Total resistance}}$

$I = \frac{120}{57.143} = 2.10A$

Thus the ammeter reading will be 2.10 amperes.

From Ohm's law;

$V_1 = IR$

$V_1 = 2.10(17.143)$

$V_1 = 36V$

Thus the voltmeter reading will be 36 volts.

From Ohm's law;

$V = IR$

$I = \frac{V}{R}$

$I_1 = \frac{V}{R_1} = \frac{36}{30} = 1.2A$

$I_2 = \frac{V}{R_2} = \frac{36}{60} = 0.6A$

$I_3 = \frac{V}{R_1} = \frac{36}{120} = 0.3A$

Ammeter reading $= I_2 + I_3$

Ammeter reading $= 0.6 + 0.3$

$\underline{\text{Ammeter reading} = 0.9A}$

(iii)

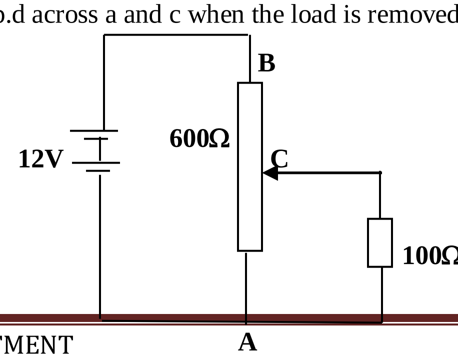
$\% \, change; = \frac{Change \; in \; current}{Original \; value} \times 100\%$

$\% \, change; = \left(\frac{I - I_1}{I}\right) \times 100\%$

$\% \, change; = \left(\frac{2 - 0.9}{2}\right) \times 100\%$

$\underline{\% \, change; = 55\%}$

3. a 12V battery is connected across a potential divider of resistance 600Ω as shown below. If the load of 100Ω is connected across the terminals a and c when the slider is half way up the divider, find:
(i) p.d across the load
(ii) p.d across a and c when the load is removed.



Effective resistance

$R = \frac{300 \times 100}{300 + 100} + 300 = 375\Omega$

Current supplied by the battery, I; From Ohm's law

$V = IR$

$12 = 375I$

$I = 0.032A$

This the current through parallel combination of resistors

P.d across parallel combination of resistors, $V = 0.032 \times 75 = 2.4V$

Hence the p.d across the load is 2.4V.

(ii) When the load is removed; Effective resistance = 600Ω.

From Ohm's law;

$V = IR$

$12 = 600I$

$I = 0.02A$

Hence p.d across AC is given by;

$$V_{AC} = 0.02 \times 300 = 6V$$

3. The circuit below, the battery has negligible internal resistance. Find the ammeter and voltmeter reading.

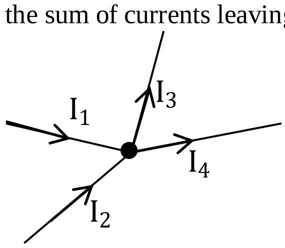
## KIRCHHOFF'S LAWS

These are two laws:

**1st law:** This law is also referred to as the law of conservation of current at a junction.

It is derived from the fact that charges do not accumulate at a point.

It states that the sum of currents entering a junction is equal to the sum of currents leaving the junction.




From Kirchhoff's 1st law;

$\mathbf{I_1 + I_2 = I_3 + I_4}$

**OR:**

It states that the algebraic sum of all currents at a junction is zero.

Here all currents entering a junction are given a positive sign and those leaving the junction are given a negative sign.

That is;

$\sum \mathbf{I = 0}$

$I_1 + I_2 + (-I_3) + (-I_4) = 0$

$I_1 + I_2 - I_3 - I_4 = 0$

**2nd law:** This law is sometimes referred to as the closed loop equation.

It states that in any closed loop, the algebraic sum of all potential drops is equal to the algebraic sum of all the E.m.fs. That is;

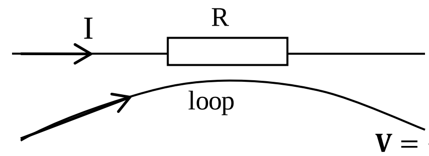
$\sum \mathbf{P.d} = \sum \mathbf{E.m.f}$

Note: A loop is a closed path.

Sign allocation to P.ds

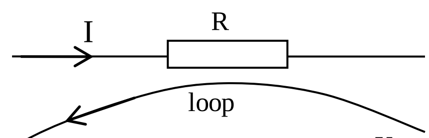
When allocating signs to P.ds across a resistor in a closed loop, the direction of current flowing through the resistor is considered in relation to the direction of the loop.

The p.d across the resistor is considered positive if the current is in the same direction as that of the loop.




$\mathbf{V = +IR}$

The p.d across the resistor is considered negative if the current is in the opposite direction to that of the loop.

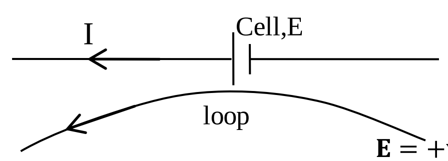



$\mathbf{V = -IR}$

Sign allocation to E.m.fs

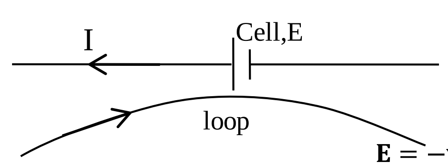
When allocating signs to E.m.f a cell in a closed loop, the polarity of the cell is considered in relation to the direction of the loop.

The E.m.f of the cell is considered positive if the loop moves from negative terminal to the positive terminal.




$\mathbf{E = +ve}$

The E.m.f of the cell is considered negative, if the loop moves from positive terminal towards the negative terminal.



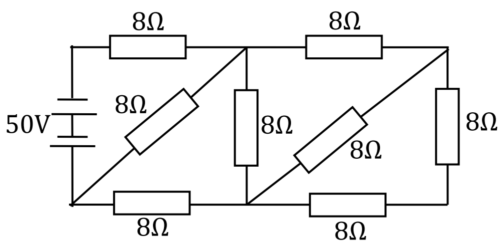

$\mathbf{E = -ve}$

**Remarks:**

In some circuit networks, the direction of current may not be given. In such cases, a student is free to choose his own convenient direction. But after the calculation, a positive value of current will imply that the actual current flows in the direction assumed in the diagram and is of that magnitude while a negative value of current imply that the actual current flows in a direction opposite to that assumed in the diagram and is of the same magnitude.

**Example:1**

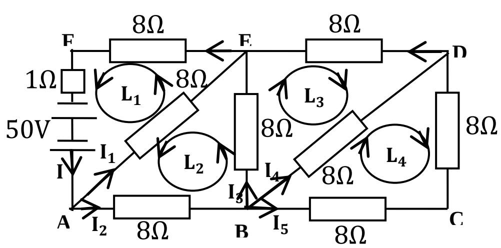
In the figure bellow, a battery X of e.m.f 50V and internal resistance of 1Ω.




Find the;

(i). Effective resistance of the circuit
(ii). Power dissipated in the battery.

**Solution**




At junction A:

$I = I_1 + I_2 \ldots\ldots\ldots\ldots\ldots\ldots (i)$

At junction B:

$I_2 = I_3 + I_4 + I_5 \ldots\ldots\ldots\ldots\ldots (ii)$

Considering loop1 ($L_1$) or Loop AEFA.

$8I_1 + 8I + I = 50$

$8I_1 + 9I = 50 \ldots\ldots\ldots\ldots\ldots (iii)$

Considering loop2 ($L_2$) or Loop ABEA.

$8I_2 + 8I_3 - 8I_1 = 0 \ldots\ldots\ldots\ldots (iv)$

Considering loop3 ($L_3$) or Loop BDCB.

$8I_3 - 8(I_4 + I_5) - 8I_4 = 0$

$8I_3 - 16I_4 - 8I_5 = 0 \ldots\ldots\ldots\ldots (v)$

Considering loop4 ($L_4$) or Loop BEDB.

$8I_4 - 8I_5 - 8I_5 = 0$

$8I_4 - 16I_5 = 0 \ldots\ldots\ldots\ldots\ldots (vi)$

From equation (v)

$8I_4 - 16I_5 = 0$

$8I_4 = 16I_5$

$I_4 = 2I_5 \ldots\ldots\ldots\ldots\ldots\ldots (vii)$

From equation (vi)

$8I_3 - 16I_4 - 8I_5 = 0$

$8I_3 - 16(2I_5) - 8I_5 = 0$

$8I_3 - 40I_5 = 0$

$8I_3 = 40I_5$

$I_3 = 5I_5 \ldots\ldots\ldots\ldots\ldots\ldots (viii)$

From equation (ii)

$I_2 = I_3 + I_4 + I_5$

$I_2 = 5I_5 + 2I_5 + I_5$

$I_2 = 8I_5 \ldots\ldots\ldots\ldots\ldots\ldots (ix)$

From equation (ii)

$8I_2 + 8I_3 - 8I_1 = 0$

$8(8I_5) + 8(5I_5) - 8I_1 = 0$

$I_1 = 13I_5 \ldots\ldots\ldots\ldots\ldots\ldots (x)$

From equation (i)

$I = I_1 + I_2$

$I = 13I_5 + 8I_5$

$I = 21I_5 \ldots\ldots\ldots\ldots\ldots\ldots (xi)$

From equation (iii)

$8I_1 + 9I = 50$

$8(13I_5) + 9(21I_5) = 50$

$I_5 = \frac{50}{293} = 0.171 \text{ A}$

$I = 21\left(\frac{50}{293}\right) = 3.584 \text{ A}$

From Ohm's law;

$V = IR_{eff}$

$50 = 3.584R_{eff}$

$\mathbf{R_{eff} = 13.952}$Ù

**ALTERNATIVELY:**

Taking path BCD;

Effective resistance $= 8 + 8 = 16\Omega$

The 16Ω and 8Ω are in parallel;

$\frac{1}{R} = \frac{1}{16} + \frac{1}{8} = \frac{3}{16} \Leftrightarrow R = \frac{16}{3}\Omega$

The $\frac{16}{3}\Omega$ and 8Ω are in series;

Effective resistance $= \frac{16}{3} + 8 = \frac{40}{3}\Omega$

The $\frac{40}{3}\Omega$ and 8Ω are in parallel along BE;

$\frac{1}{R} = \frac{3}{40} + \frac{1}{8} = \frac{8}{40} \Leftrightarrow R = \frac{40}{8} = 5\Omega$

The 5Ω and 8Ω are in series;

Effective resistance $= 5 + 8 = 13\Omega$

The 13Ω and 8Ω are in parallel;

$$\frac{1}{R} = \frac{1}{13} + \frac{1}{8} = \frac{21}{104} \Leftrightarrow R = \frac{104}{21}\,\Omega$$

Now, the $\frac{104}{21}\,\Omega$, 8Ω and 1Ω are in series;

$$\text{Effective resistance} = \frac{104}{21} + 8 + 1 = \frac{293}{21}\,\Omega$$

$$\text{Thus, Effective resistance} = \frac{293}{21}\,\Omega$$

$\mathbf{Effective\ resistance = 13.952}$Ù

**(ii)** Power dissipated in the battery.

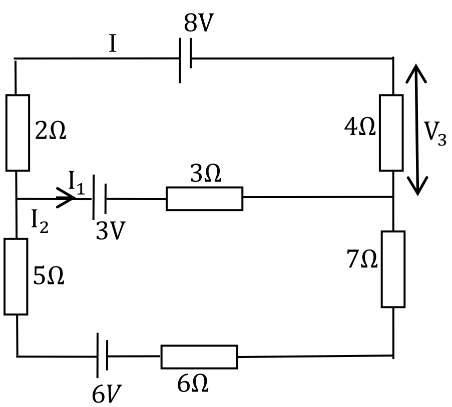
$\text{Power} = I^2 r$

$$\text{Power} = \left(\frac{1050}{293}\right)^2 (1)$$

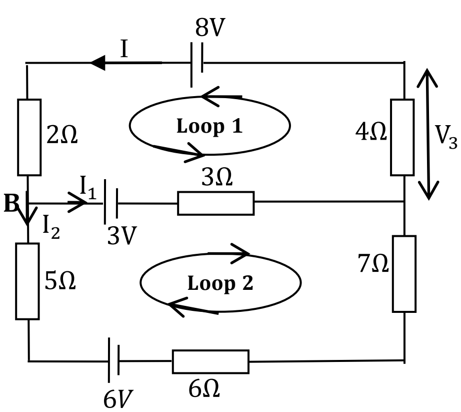
$\text{Power} = 12.842\ \text{W}$

**Example:2**

In the figure bellow, find the current flowing through the 2Ω, 3Ω and 6Ω resistors.




**Solution**




At junction B;

$I = I_1 + I_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$

Considering loop1

$2I + 3I_1 + 4I = 8 + (-3)$

$6I + 3I_1 = 5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$

Considering loop2

$3I_1 + 7(-I_2) + 6(-I_2) + 5(-I_2) = 6 + (-3)$

$3I_1 - 18I_2 = 3$

$I_1 - 6I_2 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$

Put equation (i) into equation (ii).

$6(I_1 + I_2) + 3I_1 = 5$

$9I_1 + 6I_2 = 5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (iv)$

Equation (iii) + Equation (iv):

$10I_1 = 6$

$I_1 = 0.6\ \text{A}$

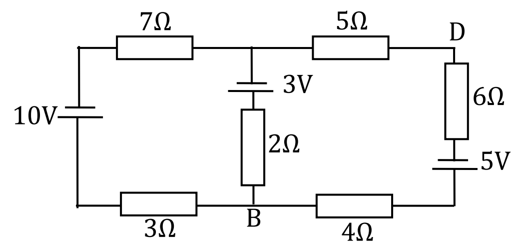
$I_2 = -0.067\ \text{A}$

($i.e\ I_2 = 0.067A\ \ in\ a\ direction\ opposite\ to\ that\ assumed\ in\ the\ diagram$).

$I\ = (0.6 + -0.067) = 0.533\ \text{A}$

**Example:3**

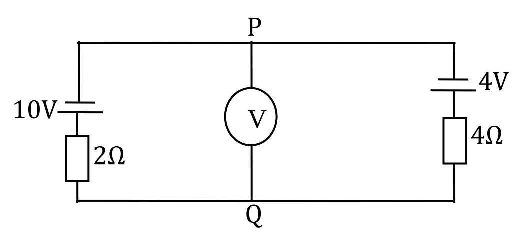
(a) In the figure bellow, find the current flowing through the 3Ω, 2Ω and 4Ω resistors.




(b) If a very high resistance voltmeter is connected across BD, what will be its reading?

**Example:1**

In the figure bellow, V is a very high resistance voltmeter. Find its reading.



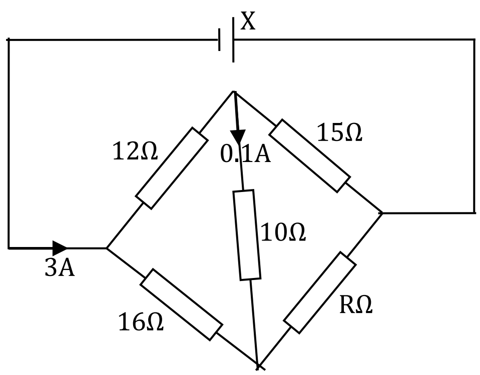

**Example:4**

In the figure bellow, use Kirchhoff's laws to find the;

(i). Current flowing through the 12Ω, 15Ω and 16Ω resistors.

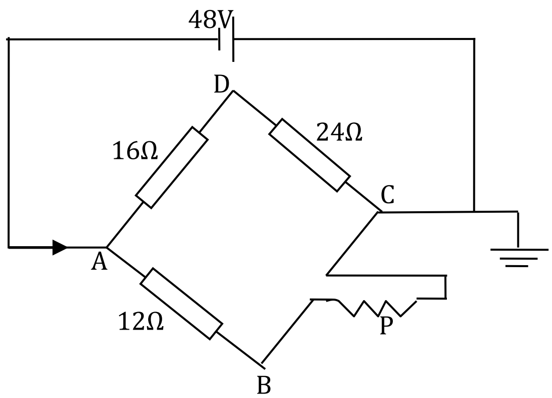
(ii). Value of R.

(iii). E.m.f of cell X.




**Example:5**

The figure bellow shows an un balanced Wheatstone bridge circuit network. P is a coil of resistance 16.5Ω at $0^0$C.




Find the;

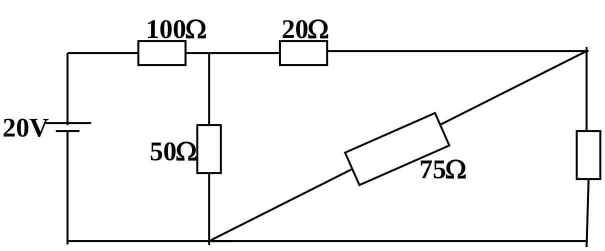
(i). Potentials at B and D.

(ii). If a galvanometer is connected across BD, in which direction would the current flow?

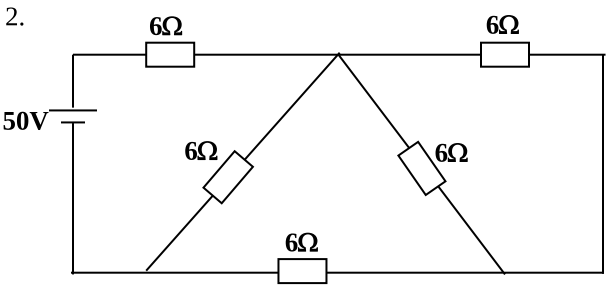
(iii). If the temperature coefficient of resistance of the coil is $2.0 \times 10^3 K^{-1}$, to what temperature must the coil be raised in order to balance?

**Exercise**

1. Find the current supplied by the battery.




2.

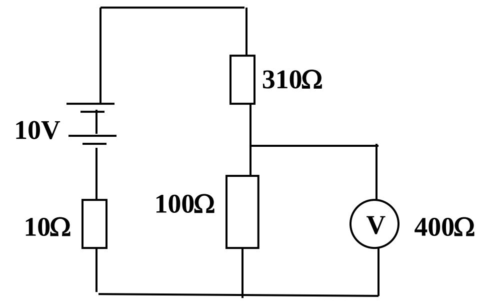



1.Given that the internal resistance of the battery in the circuit is 0.4Ω, find

(i) The effective resistance in the circuit

(ii) Power dissipated in the battery.

3.




(i) Find the reading of the voltmeter

(ii) Calculate the power dissipated in the 10Ω resistor*

4. A resistor of 500Ω and one of 200Ω are placed in series with a 6V supply. What will be the reading on a voltmeter of internal resistance 2000Ω when placed across the;

(i) 5000Ω resistor

(ii) 2000Ω resistor.

**WHEATSTONE BRIDGE**

A Wheatstone bridge circuit is an arrangement of four resistors $R_1$, $R_2$, $R_3$ and $R_4$. It used to compare resistances and to measure resistance accurately.

Circuit net work

In comparison of resistances, the unknown resistor is connected on one side of the bridge with a standard resistor on the opposite side as shown in in the circuit below.

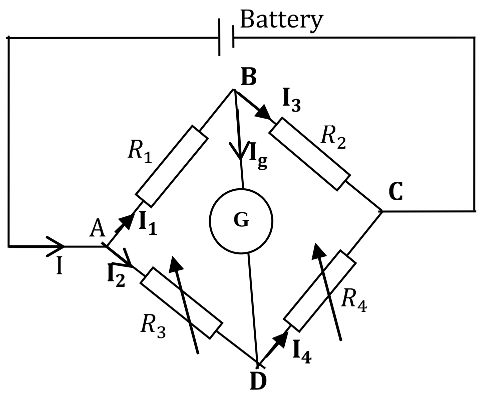
The variable resistors $\mathbf{R_3}$ and $\mathbf{R_4}$ are varied until the galvanometer shows no deflection.

As the variable resistors $R_3$ and $R_4$ are varied, the potentials at B and D change.

When the potential at B ($\mathbf{V_B}$) is greater than the potential at D ($\mathbf{V_D}$), then $\mathbf{I_g}$ flows from B to D, giving a deflection on one side of the galvanometer.

When the potential at B ($\mathbf{V_B}$) is less than the potential at D ($\mathbf{V_D}$), then $\mathbf{I_g}$ flows from D to B, giving a deflection on the opposite side of the galvanometer.

When the potential at B ($\mathbf{V_B}$) is equal to the potential at D ($\mathbf{V_D}$), then $\mathbf{I_g=0}$ **No current flows through the galvanometer.** The galvanometer shows no deflection. The bridge is said to be balanced.




G is centre zero galvanometer
$R_1$ is unknown resistor
$R_2$ is standard resistor (known resistance)
$I_g$ current flowing through the galvanometer.
$R_3$ and $R_4$ are variable resistors where resistance can be read e.g. resistance box.

As $R_3$ and $R_4$ are varied, the potential at B and D change.

(i) If the potential at B, $V_B$ is greater than that at D, $V_D$, then current $i_g$ flows from B to D giving a deflection on one side of the galvanometer.

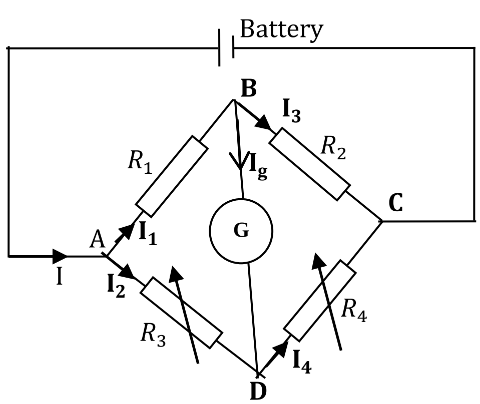
(ii) If $V_D > V_B$, $i_g$ flows from D to B, galvanometer flows in the opposite direction.

(iii) If $V_D = V_B$, $i_g = 0$, galvanometer indicates zero deflection. In this case the Wheat stone bridge is said to be balanced.

Therefore to balance the bridge, $R_3$ and $R_4$ are varied until the galvanometer indicates zero deflection.

**Conditions for the Wheatstone bridge circuit to balance**
The condition for balance can be derived in two ways:
(i). By considering the physical principles involved.
(ii). By applying the Kirchhoff's laws.

Consider the Wheatstone bridge circuit network below.




**(i)** By considering the physical principles involved.

At balance, $i_g = 0$, hence $V_D = V_B$
Hence
$V_{AB} = V_{AD}$
$I_1R_1 = I_2R_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (i)$

Similarly,
$V_{BC} = V_{DC}$
$I_1R_2 = I_2R_4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (ii)$

Equation (i) divide by equation (ii).

$$\boldsymbol{\frac{R_1}{R_2} = \frac{R_3}{R_4}}$$

The above equation is the condition for a Wheatstone bridge to balance.

**(ii)** By applying the Kirchhoff's laws.
Considering loop ABDA:
$I_1R_1 + (0)R_g - I_2R_2 = 0$
$I_1R_1 = I_2R_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (i)$

Considering loop BCDB:
$I_1R_2 - (0)R_g - I_2R_4 = 0$
$I_1R_2 = I_2R_4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (ii)$

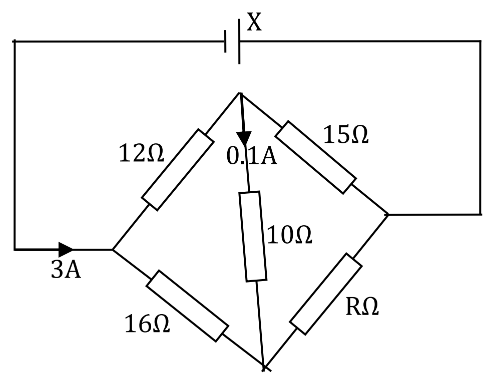
Equation (i) divide by equation (ii).

$$\frac{R_1}{R_2} = \frac{R_3}{R_4}$$

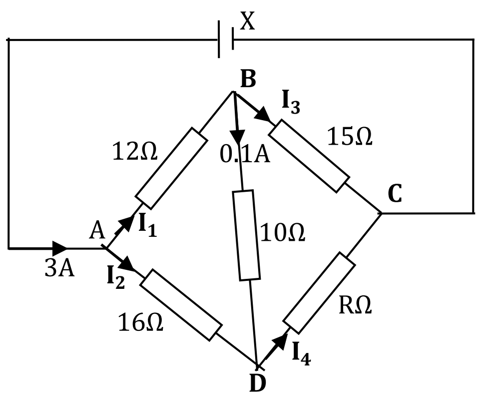
**NOTE:**
In case of measuring resistance, the un known resistance, $\mathbf{R_1}$can now be obtained from the balance condition.

$$R_1 = \frac{R_3}{R_4} \times R_2$$

**Example:1**
In the figure bellow, use Kirchhoff's laws to find the;
(i). Current flowing through the 12Ω, 15Ω and 16Ω resistors.
(ii). Value of R.
(iii). E.m.f of cell X.




**Solution:**




At junction A:
$I_1 + I_2 = 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(i)$
At junction B:
$I_3 + 0.1 = I_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$
At junction D:
$I_2 + 0.1 = I_4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(iii)$

Considering loop1 (ABDA):
$12I_1 + 10(0.1) - 16I_2 = 0$
$12I_1 - 16I_2 = -1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots(iv)$

Considering loop2 (BCDB):
$15I_3 - RI_4 - 10(0.1) = 0$
$15I_3 - RI_4 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(v)$

Considering loop3 (AXCBA):
$12I_1 + 15I_3 = X \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(vi)$

From Equation (i);
$I_2 = 3 - I_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(vii)$
Put eqn. (vii) into Eqn.(iv)
$12I_1 - 16(3 - I_1) = -1$
$28I_1 = 47$
$\underline{I_1 = 1.679\ A}$
$I_2 = 3 - 1.679 = 1.321\ A$

From equation (ii)
$I_3 = I_1 - 0.1$
$I_3 = 1.679 - 0.1 = 1.579A$

From equation (iii)
$I_4 = I_2 + 0.1$
$I_4 = 1.321 + 0.1 = 1.421$ A

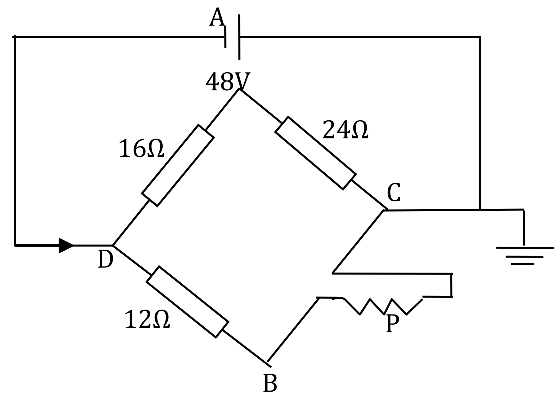
From equation (v)
$RI_4 = 15I_3 - 1$

$$R = \frac{15 \times 1.579 - 1}{1.421} = 15.964\Omega$$

From equation (v)
$12I_1 + 15I_3 = X$
$12(1.679) + 15(1.579) = X$

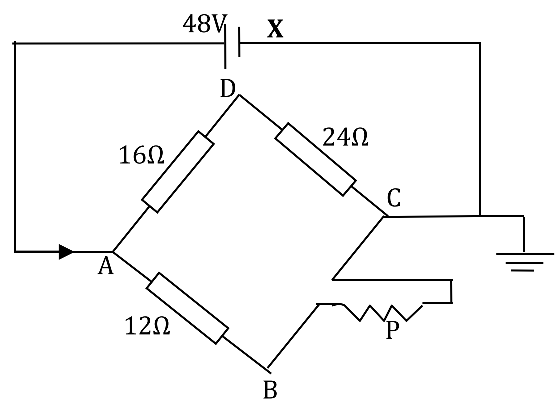
$\underline{\underline{X = 43.833V}}$

**Example:2**
The figure bellow shows an un balanced Wheatstone bridge circuit network. P is a coil of resistance 16.5Ω at $0^0$C.




Find the;
(i). Potentials at B and D.
(ii). If a galvanometer is connected across BD, in which direction would the current flow?
(iii). If the temperature coefficient of resistance of the coil is $2.0 \times 10^3 K^{-1}$, to what temperature must the coil be raised in order to balance?

**Solution:**




Considering loop1 (AXCDA):
$16I_1 + 24I_1 = 48$
$40I_1 = 48$
$\mathbf{I_1 = 1.2A}$

Considering loop2 (AXCDA):
$12I_2 - 16.5I_2 = 48$
$128.5I_2 = 48$
$I_2 = 1.684\ A$

Since current flows from a region of high potential to that of low potential,
$V_{CB} = V_B - V_C = IR$
$But, V_C = 0$
$(since potential due to the earth is zero)$

Thus;
$V_{CB} = V_B = IR$
$V_{CB} = V_B = 1.684 \times 16.5 = 27.79V$
$\underline{\underline{V_{CB} = V_B = 27.79V}}$

$V_{CD} = V_D - V_C = IR$
$But, V_C = 0$
$(since potential due to the earth is zero)$
Thus;
$V_{CD} = V_D = IR$
$V_{CD} = V_D = 1.2 \times 24 = 28.8V$
$\underline{\underline{V_{CD} = V_D = 28.8V}}$

(ii) The current would flow from D to B since D is at a higher potential than B.

(iii) Let the temperature be è;

At balance,
$\frac{R_è}{12} = \frac{24}{16} \Leftrightarrow R_è = \frac{24}{16} \times 12 = 18\Omega$

Using the equation:
$R_è = R_0(1 + áè)$

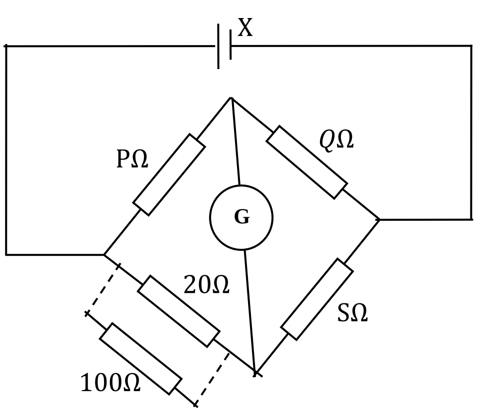
$á = 2.0 \times 10^{-3}K^{-1}, R_0 = 16.5\Omega$

$\Leftrightarrow 18 = 16.5(1 + 2.0 \times 10^{-3}è)$
$\underline{\underline{\Leftrightarrow è = 45.5^0C}}$
Thus the coil must be raised to $45.5^0C$

**Questions**

(i). Draw a Wheatstone bridge circuit network and use it to derive the balance condition. Explain how the network can be used in practice to measure resistance.

(ii). In the, the $20\Omega$ resistor is shunted with $100\Omega$ resistor for a bridge to balance. When P and Q are interchanged, the shunt resistance changes to $50\Omega$ for the bridge to balance again.



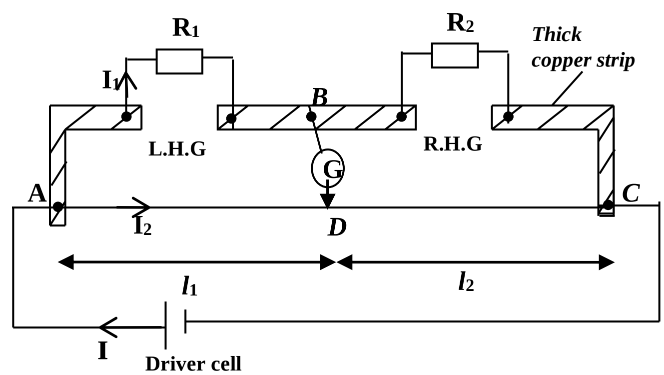

Find the
  (i). Value of S.[Ans:S=15.43Ω]
  (ii). value of the true ratio of P:Q.+-[Ans;=1.08]

(iii). In a Wheatstone bridge, the ratio arms $R_1$ and $R_2$ are approximately equal. When $R_3 = 500\Omega$, the bridge is balanced. On interchanging $R_1$ and $R_2$, the value of $R_3$ for balancing is $505\Omega$. Find the value of $R_4$ and the ratio $R_1 : R_2$. ($502.5\Omega$, 1:1.005)

## THE METRE BRIDGE

It is a form of a Wheatstone bridge with resistance slide wire of uniform cross section area mounted on a metre rule. It is connected in series with an accumulator (Driver cell) which maintains a steady current through the wire.
A metre-bridge is used to measure resistance, resistivity and temperature coefficient of resistance.

Circuit Network of a metre bridge.




**Procedures:**
The un known resistor is connected in the left hand gap (L.H.G) and the standard resistor in the right hand gap (R.H.G) of the metre bridge.
The jokey is placed at different points along the uniform slide wire until the galvanometer shows no deflection. The metre bridge is said to be balanced.
The balance lengths, $l_1$ and $l_2$ are measured and recorded.
The un known resistance, is then measured from;
$\frac{R_1}{R_2} = \frac{l_1}{l_2}$

**Condition for a metre bridge to balance.**
Consider the metre- bridge circuit network above.
At balance, $i_g = 0$, hence $V_D = V_B$
Hence
$V_{AB} = V_{AD}$
$I_1R_1 = I_2R_{AD} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (i)$

Similarly,
$V_{BC} = V_{DC}$
$I_1R_2 = I_2R_{DC} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (ii)$

Equation (i) divide by equation (ii).
$\frac{R_1}{R_2} = \frac{R_{AD}}{R_{DC}}$
But,

$R_{AD} \propto l_1 \Leftrightarrow R_{AD} = kl_1$
$R_{DC} \propto l_2 \Leftrightarrow R_{DC} = kl_2$
Where, k is the resistance per centimeter if $l_1$ and $l_2$ are measured in centimeters.

$$\frac{R_1}{R_2} = \frac{l_1}{l_2}$$

**Alternatively;**

, $R_{AD} = ñ\frac{l_1}{A}$where A is the cross sectional area of the uniform wire of the metre bridge, ρ is the resistivity of the material of wire, $l_1$ is the balance length of the metre bridge from the left hand side.

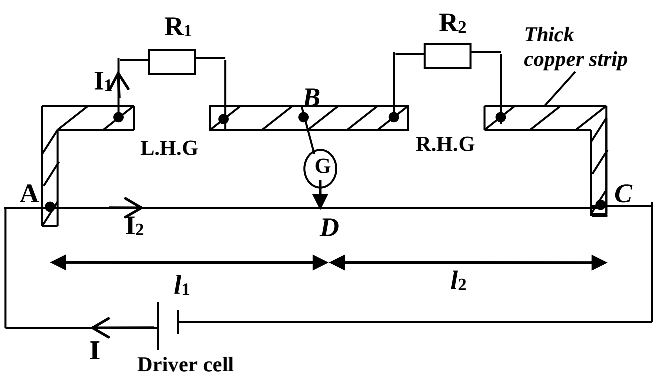
$R_{DC} = ñ\frac{l_2}{A}$, where $l_2$ is the balance length of the metre bridge from the right hand side.

Hence: $\boldsymbol{\frac{R_1}{R_2} = \frac{l_1}{l_2}}$

The above equation is the condition for a metre bridge to balance.

If the length of the wire is 100cm then $l_2 = (100 - l_1)$ cm.

**End errors or end corrections**




Owing to imperfect electrical contacts at A and C, contacts have small resistance which affect the accuracy of the results. But the effect on the accuracy becomes less significant if the balance point is near the 50cm mark.

For accuracy, the contact resistances have to be considered. The contact resistances at A and C are equivalent to extra lengths $e_1$ and $e_2$ of the slide wire. $e_1$ and $e_2$ are called end corrections or end errors.

Hence end errors have to be added to the balance lengths to account for the resistances at the contacts at the ends of the slide wire.

At balance

$$\frac{R_1}{R_2} = \frac{l_1 + e_1}{l_2 + e_2} \text{.......... .....}(i)$$

When $R_1$ and $R_2$ are interchanged

$$\frac{R_1}{R_2} = \frac{l_1^{'} + e_1}{l_2^{'} + e_2} \text{.......... ..}(ii)$$

where $l_1^{'} and l_2^{'}$ are new balance lengths from the left hand side and right hand side respectively.

Solve equations (i) and (ii) simultaneously to obtain $e_1$ and $e_2$.

**Example**

When resistors of 3Ω and 5Ω are connected in the LHD and RHG of a metre bridge respectively, a balance point is obtained at 37.4cm from LHS. When the resistors are interchanged, the balance point is 62.8cm from the LHS. The resistance of the slide wire is 10Ω. Calculate the end corrections and resistance of the contacts at LHS and RHS.

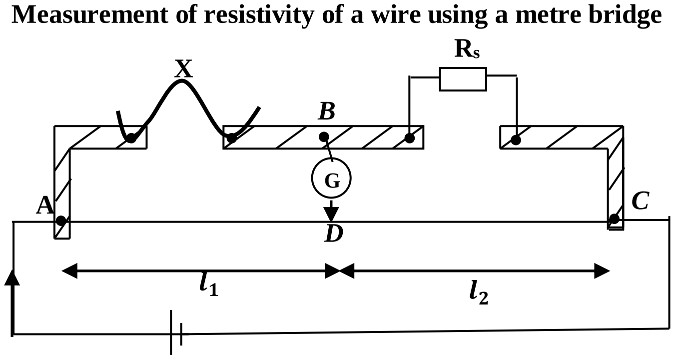
$$\frac{3}{5} = \frac{37.4 + e_1}{62.6 + e_2} \text{.......... .....}(i)$$

$$\frac{5}{3} = \frac{62.8 + e_1}{37.2 + e_2} \text{.......... .....}(ii)$$

Solve equations (i) and (ii) simultaneously you obtain $e_1$ = 0.7cm and $e_2$ = 0.9cm.

Resistances from zero end $r_1$=0.07Ω, on the right end $r_2$ = 0.09Ω.

**Measurement of resistivity of a wire using a metre bridge**




The diameter of the specimen wire is measured at different points of the wire using a micrometer screw gauge and there after the mean diameter, **d** of the wire determined. The cross sectional area **A** of the wire is then calculated from; $\mathbf{A = \frac{ðd^2}{4}}$.

A specimen wire of known measured length $\boldsymbol{x}$ is connected in the left hand gap of a metre bridge with a standard resistor, $\mathbf{R_s}$ in the right hand gap of the bridge.

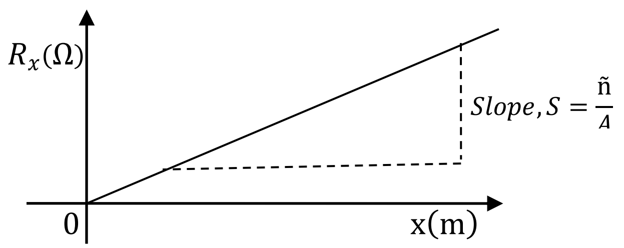
The slider is moved along AC until the galvanometer indicates zero deflection. At this point the metre bridge is balanced. The balance length $l_1$ and $l_2$ from the left and right hand side of the metre bridge are measured.

The experiment is repeated for different values of $x$ and the corresponding balance lengths $l_1$ and $l_2$ are measured and recorded.

The results are tabulated including values of $\boldsymbol{R_x = R_s\left(\frac{l_1}{l_2}\right)}$:

| $x(cm)$ | $l_1(cm)$ | $l_2(cm)$ | $R_x(\Omega)$ |
|---|---|---|---|
| - | - | - | - |
| - | - | - | - |

A graph of R against $x$ is plotted. It is a straight line graph through the origin.




The slope,S of the graph is determined and is equal to:

$$\boldsymbol{Slope, S} = \frac{ñ}{\boldsymbol{A}} = \frac{ñ}{\left(\frac{\mathbf{ðd^2}}{\mathbf{4}}\right)}$$

Hence resistivity of the wire, $ñ = \boldsymbol{Slope} \times \boldsymbol{Area} = \boldsymbol{S}\left(\frac{\mathbf{ðd^2}}{\mathbf{4}}\right)$.
Where **d** is the diameter of the wire which can be measured by a micrometer screw gauge.

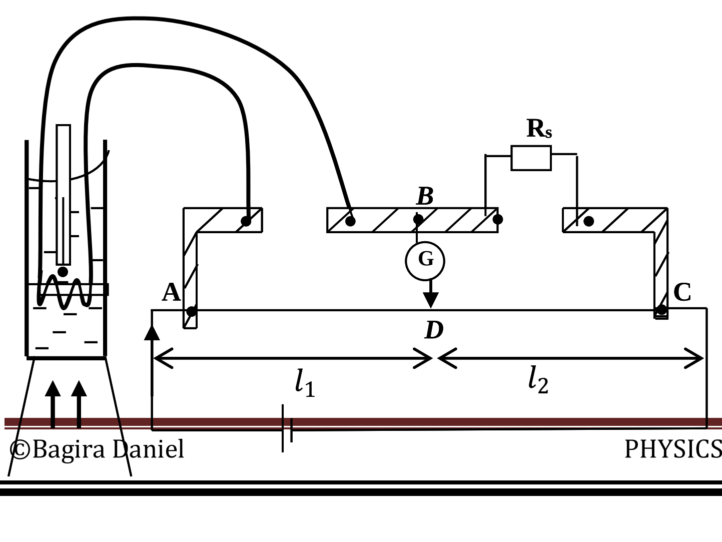
<u>Theory of experiment</u>
At balance,
$\frac{R_x}{R_s} = \frac{l_1}{l_2} \Leftrightarrow R_x = \frac{R_s l_1}{l_2} \Leftrightarrow R_x = R_s\left(\frac{l_1}{l_2}\right)$
From

$$R_x = ñ\frac{x}{A} \Leftrightarrow R_x = \left(\frac{ñ}{A}\right)x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Compare this equation with the general equation of a straight line:
$y = mx + c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (ii)$
$\Leftrightarrow m = Slope, S = \frac{ñ}{A}$
$\Leftrightarrow c = o$

**Temperature coefficient (α) of a metal wire**
The mean temperature coefficient of a material can be defined as the fractional increase in resistance at 0°C for every °C increase in temperature.
$á = \frac{\boldsymbol{R_è - R_0}}{\boldsymbol{R_0}è}$
$R_0$is resistance at 0°C, $R_\theta$ is the resistance at temperature θ°C.
Hence; $R_è = R_0(1 + áè)$.
For semi-conductors, resistance decreases as temperature increases hence α is negative.

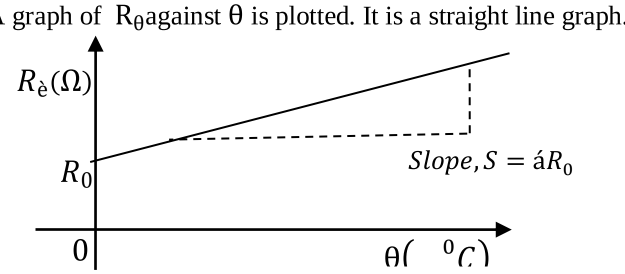
**Experiment to determine temperature coefficient of resistance using the metre bridge**




The specimen wire is made into a coil and immersed in water basin whose temperature θ°C is varied and measured with a thermometer placed in the water.
The coil is connected to the left hand gap of the bridge and a standard resistor $R_s$ to the right hand gap.
At each temperature, θ°C, The slider is moved along AC until the galvanometer indicates zero deflection. At this point the metre bridge is balanced. The balance length $l_1$ and $l_2$ from the left and right hand side of the metre bridge are measured.
The experiment is repeated for different values of temperature è and the corresponding balance lengths $l_1$ and $l_2$ are measured and recorded.
The results are tabulated including values of the resistance at a temperatureè, $\boldsymbol{R_è} = \boldsymbol{R_s}\left(\frac{\boldsymbol{l_1}}{\boldsymbol{l_2}}\right)$:

| $è(\ ^0C)$ | $l_1(cm)$ | $l_2(cm)$ | $R_è(\Omega)$ |
|---|---|---|---|
| - | - | - | - |
| - | - | - | - |

A graph of $R_\theta$against θ is plotted. It is a straight line graph.




Temperature Coefficient of resistance of the wire is;
$á = \frac{\mathbf{s}}{\mathbf{R_0}} = \frac{\mathbf{Slope}}{\mathbf{intercept}}$.

<u>Theory of experiment.</u>
From;
$R_è = R_0(1 + áè)$
$R_è = R_0 + (áR_0)è$
$R_è = (áR_0)è + R_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (i)$

Compare this equation with the general equation of a straight line:
$y = mx + c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (ii)$
$\Leftrightarrow m = Slope, S = áR_0$
$\Leftrightarrow c = R_0$

**Examples**
1. When a coil x is connected across the Left hand gap of a metre bridge and heated to a temperature of 30°C, the balance point is found to be 51.5°C from the left hand side of the slide

wire. When the temperature is raised to 100°C, the balance point is 54.6cm from the left hand side. Find the temperature coefficient of resistance of x

**Solution.**

$\frac{\boldsymbol{R_1}}{\boldsymbol{R_2}} = \frac{\boldsymbol{l_1}}{\boldsymbol{l_2}}$; where $\boldsymbol{l_2 = 100 - l_1}$. For a metre − bridge.

$$R_{30} = \frac{51.5R_s}{100 - 51.5} = 1.06R_s$$

$$R_{100} = \frac{54.6R_s}{100 - 54.6} = 1.203R_s$$

But;

$\boldsymbol{R_{\text{è}} = R_0(1 + \text{èá})}$

$\Leftrightarrow\ R_{30} = R_0(1 + 30\text{á}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$

$\Leftrightarrow R_{100} = R_0(1 + 100\text{á}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (ii)$

Equation (i) divided by Equation (ii) gives;

$$\frac{R_{30}}{R_{100}} = \frac{1.06R_s}{1.203R_s} = \frac{R_0(1 + 30\text{á})}{R_0(1 + 100\text{á})}$$

$\text{á} = \boldsymbol{2.01 \times 10^{-3} K^{-1}}$

**Note:**

The Wheatstone bridge can measure accurately resistances from 1Ω to $10^6$Ω . It cannot be used to measure resistances less than 1Ω because the contact resistances become comparable to the test resistances. The bridge cannot be used to measure accurately the resistance above $10^6$Ω because the galvanometer becomes insensitive.

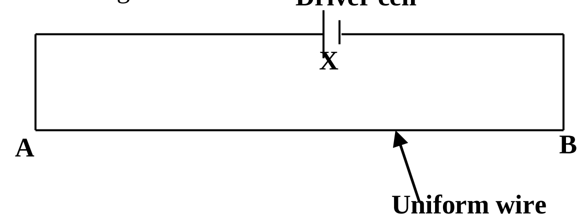
**Exercise**

1. Two resistance coils P and Q are placed in the gaps of a metre bridge. A balance point is found when the movable contact touches the bridge wire at a distance of 35.5cm from the end joined to end P. When the coil Q is shunted with a resistance of 10Ω, the balance point is moved through a distance of 15.5cm. Find the values of the resistances P and Q.
2. In a metre bridge when a resistance in left gap is 2Ω and unknown resistance in right gap, the balance point is obtained from the zero end at 40cm on the bridge wire. On shunting the unknown resistance with 2Ω, find the shift of the balance point on the bridge wire. (22.5cm)
3. With a certain resistance in the left gap of a slide wire, the balancing point is obtained when a resistance of 10Ω is taken out from the resistance box. On increasing the resistance from the resistance box by 12.5Ω, the balancing point shifts by 20cm. Find the value of unknown resistance. (15Ω)
4.

## POTENTIOMETERS

Potentiometers are used to measure p.d. As a result of measuring resistance and current can also be obtained.

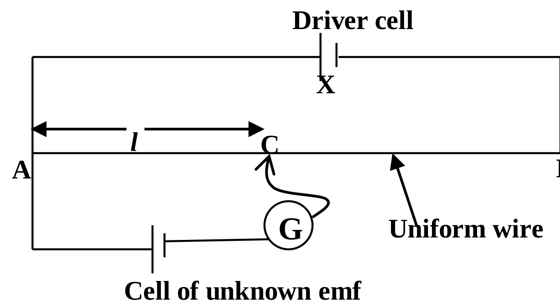
Potentiometer consists of a uniform wire of length 1m long mounted on a metre rule. A driver cell is used to maintain a current through the wire.




**Principle of a slide wire potentiometer.**

- The unknown p.d to be determined is connected in opposition to the driver cell.

The un known P.d could be the p.d across a resistor, terminal p.d of a cell on either open or closed circuit, or e.m.f of a thermocouple.



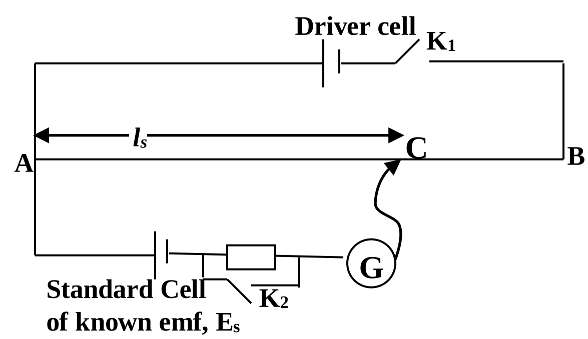

- The slide (jockey) is moved along a slide wire until a point is reached where the galvanometer does not deflect. At this point the potentiometer is said to be balanced.
- At balance, the p.d across the balance length is equal to the p.d of unknown cell. Hence emf of cell is equal to the pd across the balance length.
- But p.d across balance length is proportional to balance length.

$V_{AC} \propto l$

$\boldsymbol{V_{AC} = kl}$

Where $l$ is balance length, k is constant called p.d per unit length. If $l$ is in cm, then k = p.d/cm.

**Calibration/ standardization of a potentiometer.**




- Switch $K_1$ is closed and the circuit checked for a two way deflection by moving the slide contact towards A and B. G should deflect in opposite direction.

- The sliding contact is moved along AB until the galvanometer shows no deflection and the balance length is approximately located.
- Switch K2 is closed and the point located accurately using the full sensitivity of the galvanometer. The balance length $l_s$ is measured and recorded.
- At balance;$E_s = Kl_s$

Where K is the p.d per cm of the slide wire and it constitutes the calibration constant of the potentiometer.

The calibration constant $\mathbf{K} = \frac{\mathbf{E_s}}{\mathbf{l_s}}$

**NOTE:**

If the resistance of the wire is known, then we can calculate the steady current flowing in the slide wire given the e.m.f of the standard cell.

At balance;$l_{AC}$

Let I be the steady current flowing.

$E_s = V_{AC} = Kl_{AC} \dots\dots\dots\dots\dots\dots.(i)$

Let the resistance per centimeter be (R/cm)

$R_{AC} = (R/cm)l_{AC}$

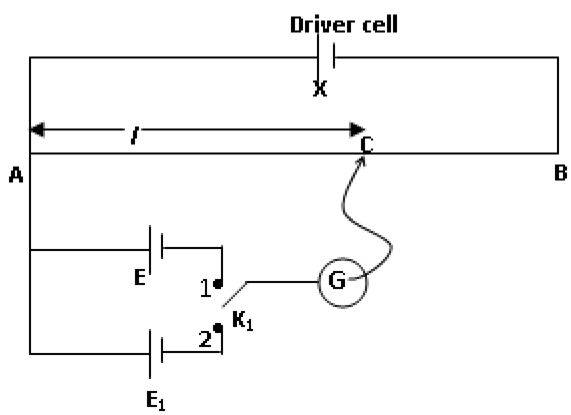
$E_s = V_{AC} = IR_{AC}$

$E_s = V_{AC} = I(R/cm)l_{AC} \dots\dots\dots.(ii)$

Hence the current I can be calculated from;

$$I = \frac{E_s}{(R/cm)l_{AC}}$$

**Measuring emf/comparing emf using a potentiometer**.




- Switch $K_1$ is connected to position 1 and the sliding contact (jockey) is moved along the wire AB until the galvanometer indicates zero deflection. The corresponding balance length $l$ is measured.

At balance, $E = kl$.......... ..$(i)$

- $K_1$ is then connected to position 2, the jockey is moved along the slide wire AB until the galvanometer indicates zero deflection, and the corresponding balance length $l_1$ is measured.

At balance $E_1 = kl_1$.......... ..$(ii)$

Equation (i) divide it by equation (ii)

$$\frac{E}{E_1} = \frac{l}{l_1}$$

Hence the e.mfs can be compared. In case of measurement of of the e.m.f., E of the cell;

$$\boldsymbol{E = \left(\frac{l}{l_1}\right) \times E_1}$$

***Advantages of potentiometer over a moving coil voltmeter.***

(i) The potentiometer is more accurate because it does not draw current from the circuit whose p.d it is meant to measure. The potentiometer can be considered to be a voltmeter with an infinitely high resistance which is an ideal voltmeter.

(ii) The potentiometer method is a null method. The accuracy of the potentiometer does not depend on the accuracy of the galvanometer but only on its sensitivity. The accuracy of the result is not affected by the fault accuracy of the galvanometer.

(iii) It has a wide range of only limited by the value of the e.m.f of the driver cell.

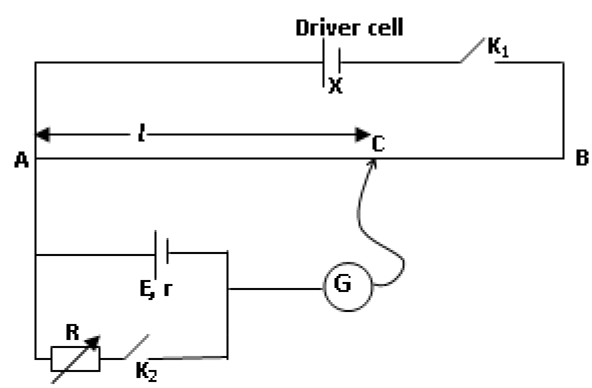
***Disadvantages of potentiometer over a moving coil voltmeter***

(i) It does not give direct reading

(ii) It requires a skilled person

(iii) It is slow in operation.

**Measuring internal resistance using potentiometer**




- Switch $K_1$ is closed. A jockey is moved along wire AB until the galvanometer indicates zero deflection. The balance length,$l_0$is measured and recorded.

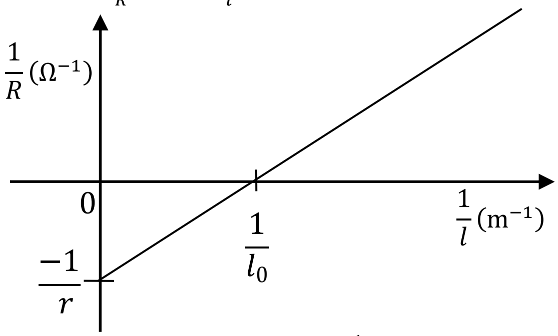
At balance; $E = kl_0 \dots\dots\dots\dots\dots\dots\dots(i)$

- Switch $K_2$ is then closed with R set to suitable value of resistance. The balance point where the galvanometer indicates zero deflection is obtained.
- The corresponding balance length $l$ is measured and recorded.

At balance; $IR = kl$ ……………………$(ii)$

- The experiment is repeated for different settings of resistance, R and each time the corresponding balance length $l$ is measured. The results are recorded in a suitable table including values of $\frac{1}{R}$ and $\frac{1}{l}$.

A graph of $\frac{1}{R}$ against $\frac{1}{l}$ is plotted.




- The negative intercept on the $\frac{1}{R}$-axis is read and recorded.
- The internal resistance, r of the cell is then obtained from;

$$\mathbf{r} = \frac{-\mathbf{1}}{\textbf{Intercept on the } \frac{\mathbf{1}}{\mathbf{R}} - \textbf{axis}}$$

**Theory of experiment**

When $K_1$ is closed and $K_2$ open,

At balance; $E = kl_0$ ……………………$(i)$

When both $K_1$ and $K_2$ closed;

At balance; $IR = kl$

$$\text{But}, I = \frac{E}{(R + r)}\ ; \text{Hence } IR = \frac{ER}{(R + r)}$$

$$\Leftrightarrow \quad \frac{ER}{(R + r)} = kl \text{ ……………………} (ii)$$

Equation (i) divide by (ii)

$$\frac{E}{\left[\frac{ER}{(R+r)}\right]} = \frac{l_0}{l} \Leftrightarrow \frac{(R + r)}{R} = \frac{l_0}{l}$$

$$\Leftrightarrow 1 + \frac{r}{R} = \frac{l_0}{l}$$

$$\Leftrightarrow \frac{r}{R} = \frac{l_0}{l} - 1$$

$$\Leftrightarrow \frac{\boldsymbol{1}}{\boldsymbol{R}} = \left(\frac{\boldsymbol{l_0}}{\boldsymbol{r}}\right)\frac{\boldsymbol{1}}{\boldsymbol{l}} - \frac{\boldsymbol{1}}{\boldsymbol{r}} \text{ ……………………} (\boldsymbol{iii})$$

Compare this equation with the general equation of a straight line:

$$y = mx + c \text{ ………………………………} (iv)$$

$$\Leftrightarrow \boldsymbol{m} = Slope, \boldsymbol{S} = \frac{\boldsymbol{l_0}}{\boldsymbol{r}}$$

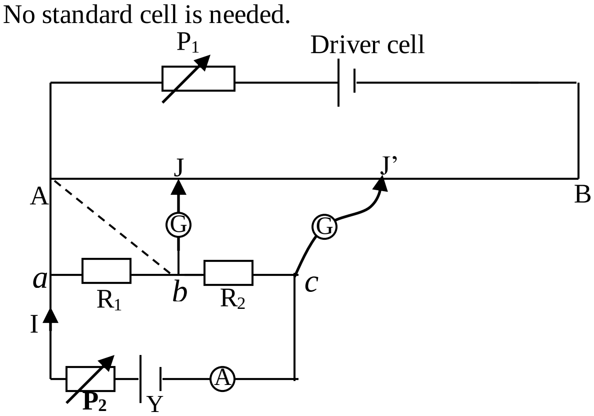
$$\Leftrightarrow \boldsymbol{C} = \frac{1}{R} - axis\ intercept = -\frac{\boldsymbol{1}}{\boldsymbol{r}}$$

**Comparison of resistance using a potentiometer**

Potentiometers can be used to compare resistances by comparing the potential difference across them when they are carrying the same current.

This method is very useful for very low resistances because the resistance of the connecting wires do not affect the results of the experiment. It can still be used for higher resistances.

However, with low resistances, an ammeter, A and a rheostat $P_2$ are necessary to adjust the current to a value that will neither exhaust the accumulator, Y nor over heat the resistors. No standard cell is needed.




- The two resistors to be compared are connected in series so that the same current flows through them. With the galvanometer at ***a*** and ***b***, the jokey is placed at different points along the wire until the galvanometer shows no deflection. The balance length AJ = $l_1$ is measured and recorded. **At balance, P.d across $R_1$ = P.d across J**

$$IR_1 = kl_1 \text{…………..(i)}$$

- Connections at ***a*** and ***b*** are removed and replaced by those at ***b*** and ***c***(dotted lines). The p.d across $R_2$ is also balanced against the slide wire.
- The balance length AJ' = $l_2$ is measured and recorded.

**At balance, P.d across $R_1$ = P.d across J**

$$IR_2 = kl_2 \text{………………(ii)}$$

Equation (i) divide by equation (ii) gives.

$$\frac{\boldsymbol{R_1}}{\boldsymbol{R_2}} = \frac{\boldsymbol{l_1}}{\boldsymbol{l_2}}$$

Hence two resistors can be compared.

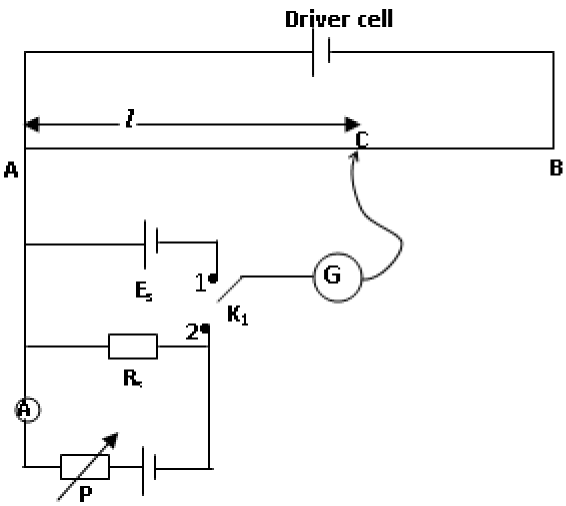
***Advantages of Potentiometers over Wheatstone bridge***

(i). In potentiometer method, no current flows through the connecting wires at balance un like in the Wheatstone method.

(ii). In calculations, the resistance of the connecting wires is not required.

(iii). It is very suitable in comparison of very low resistances.

**Calibration of Ammeter using potentiometer**




Switch $K_1$ is connected to position 1. The jokey is placed at different points along the wire until the galvanometer shows no or zero deflection. The balance length $l_s$ is measured and recorded.

$E_s = Kl_s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$

Switch $K_1$ is then connected to position 2. The rheostat P is adjusted so that the ammeter records the smallest current $\boldsymbol{I_r}$. . The jokey is placed at different points along the wire until the galvanometer shows no or zero deflection. The balance length $l$ is measured and recorded.

$I_aR_s = kl \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$

The experiment is repeated for different adjustments of the rheostat P and hence for different readings of the ammeter$\boldsymbol{I_r}$. Balance length $l$ is determined in each case and the results tabulated including values of $\boldsymbol{I_a = \frac{E_sl}{R_sl_s}}$.

| $I_r(A)$ | $l(cm)$ | $I_a(A)$ |
|---|---|---|
| - | - | - |
| - | - | - |

A graph of $\boldsymbol{I_a}$ against $\boldsymbol{I_r}$ is plotted and constitutes a calibration curve for the ammeter.

.

***Theory of experiment***

When $K_1$ is in position 1,

$E_s = Kl_s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$

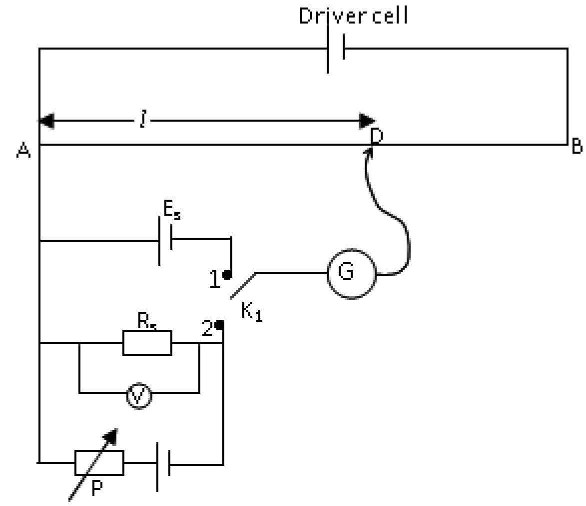
When $K_1$ is in position 2,

$I_aR_s = kl \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$

Equation (ii) divide by equation (i) & making $I_a$ the subject:

$\boldsymbol{I_a = \frac{E_sl_s}{R_sl}}$.

**Calibration of Voltmeter using potentiometer**




Switch $K_1$ is connected to position 1. The jokey is placed at different points along the wire until the galvanometer shows no or zero deflection. The balance length $l_s$ is measured and recorded.

$E_s = Kl_s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$

Switch $K_1$ is then connected to position 2. The rheostat P is adjusted so that the voltmeter records the smallest P.d it is possible to read, $\boldsymbol{V_r}$.The jokey is placed at different points along the wire until the galvanometer shows no or zero deflection. The balance length $l$ is measured and recorded.

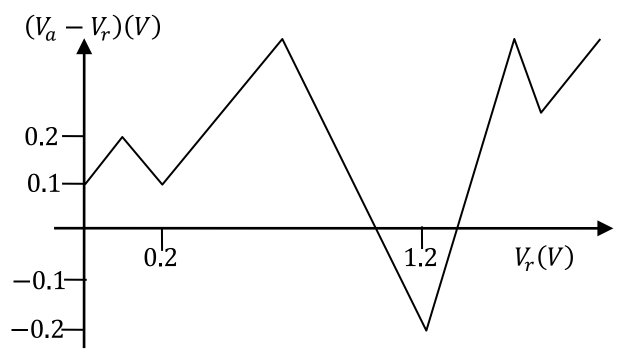
$V_a = kl \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$

The experiment is repeated for different adjustments of the rheostat P and hence for different readings of the ammeter$\boldsymbol{V_r}$. Balance length $l$ is determined in each case and the results tabulated including values of $\boldsymbol{V_a = \frac{E_sl}{l_s}}$.

| $V_r(V)$ | $l(cm)$ | $V_a(V)$ |
|---|---|---|
| - | - | - |
| - | - | - |

A graph of $\boldsymbol{V_a}$ against $\boldsymbol{V_r}$ is plotted and constitutes a calibration curve for the voltmeter.

Alternatively, a graph of $(\boldsymbol{V_a}-\boldsymbol{V_r})$against$\boldsymbol{V_r}$is plotted.




$When V_r = 0.2, the\ actual\ voltage, V_a = 0.2 + 0.1 = 0.3V$

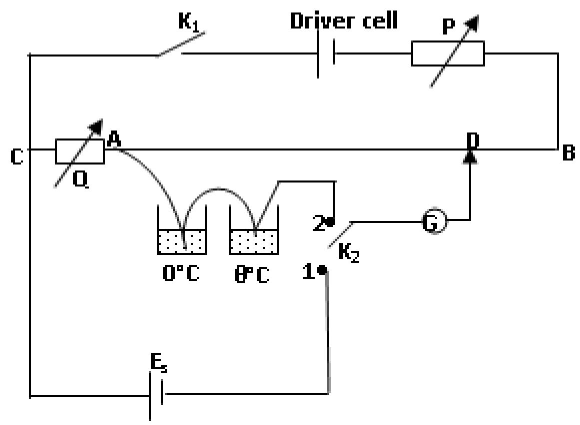
$When V_r = 1.2, the\ actual\ voltage, V_a = 1.2 + -0.2 = 1.0V$

***Theory of experiment***
When $K_1$ is in position 1,
$E_s = Kl_s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$

When $K_1$ is in position 2,
$I_a R_s = kl \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$

Equation (ii) divide by equation (i) & making $I_a$ the subject:
$\boldsymbol{V_a = \frac{E_s l}{l_s}}$.

**Measurement of small emf (e.g. emf of a thermocouple)**
Thermocouple e.m.f's are of the order of several mV (i.e of about $10^{-3}V\ or\ 10^{-4}V$. Hence it is very small to cause any considerable balance length.
In order to measure emf of a thermocouple, a slide wire AB of the potentiometer is connected in series with high variable resistance, Q. This resistor Q reduces the current flowing through AB and hence the P.d across AB to the order of that of the thermocouple. This enables the experimenter to obtain reasonable balance lengths with the thermocouple.




**Procedures.**
The variable resistors P and Q are adjusted so that the current flowing through the slide wire is very small and consequently the P.d across AB is of the same order as the thermoelectric e.m.f.

The standard cell of emf $E_s$ is connected across **Q** and the slide wire. $K_1$ is closed and $k_2$ is connected to position **1**. Keeping (P+Q) constant. The jokey is placed at different points along the wire AB until the galvanometer shows no or zero deflection.. For accuracy AD >20cm. The balance length $l_s$ is measured and recorded.
$E_s = Kl_s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$

While $K_1$ is closed, $K_2$ is connected to position **2**. The jokey is placed at different points along the wire AB until the galvanometer shows no or zero deflection i.e. When the galvanometer registers zero current is found. The balance length *l* is measured and recorded.
$E_T = Kl \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(ii)$

The e.m.f of the thermo couple is given by;-
$\boldsymbol{E_T = \frac{E_s l}{l_s}}$.

Alternatively, the e.m.f of the thermo couple is calculated from;
$$\boldsymbol{E_T = \frac{E_s(rl)}{[Q + (rl_s)]}}$$
Where r is the resistance per cm of the slide wire.

**Theory of experiment**
Let the Current through the wire AB, be I.
When $k_2$ is connected to position **1**, the standard cell is balanced against the slide wire CB. Hence At balance,
$E_s = V_{CD} = I(Q + rl_s) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$
Where, r is the resistance per centimeter of the slide wire AB.

When $k_2$ is connected to position **2**, the thermocouple is balanced against the slide wire CB. Hence At balance,
$E_T = V_{AD} = I(rl_s) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(ii)$
Where, r is the resistance per centimeter of the slide wire AB.

Equation (ii) divide equation (i) gives

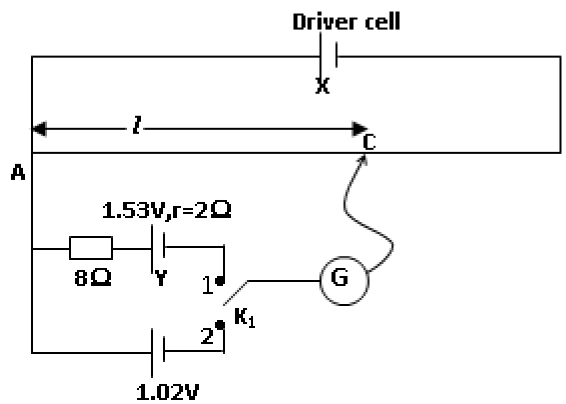
$$\boldsymbol{E_T = \frac{E_s(rl)}{[Q + (rl_s)]}}$$

***Examples***
1) The emf of a battery Y is balanced by a length of 75cm on a potentiometer wire. The emf of the standard cell, 1.02V is balanced by a length of 50cm.
(i) What is the emf of Y.

**Solution.**
When $k_1$ is placed to position 2, At balance
$1.02 = 50K \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$
$K = 0.0204Vcm^{-1}$

When $k_1$ is placed to position 1, At balance
$E = 75K \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(ii)$
$E = 0.0204 \times 75$
$\boldsymbol{E = 1.53V}$

(ii) Calculate the new balance length if Y has internal resistance of 2Ω and a resistor of 8Ω is joined to the positive terminal of Y.




$E = I(R + r)$
$1.53 = I(8 + 2)$
$I = 0.153A$

$V_{AC} = Kl$
$IR = Kl \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(iii)$
$0.153(8) = 0.0204l$
$\boldsymbol{l = 60cm}$

Alternatively:
$\frac{E}{V} = \frac{l}{l_1}$
$\frac{1.53}{V} = \frac{75}{l_1}, Where V = IR = \frac{ER}{R + r} = \frac{1.53 \times 8}{8 + 2} = 1.224V$

$\frac{1.53}{1.224} = \frac{75}{l_1}$

$\boldsymbol{l_1 = 60cm}$

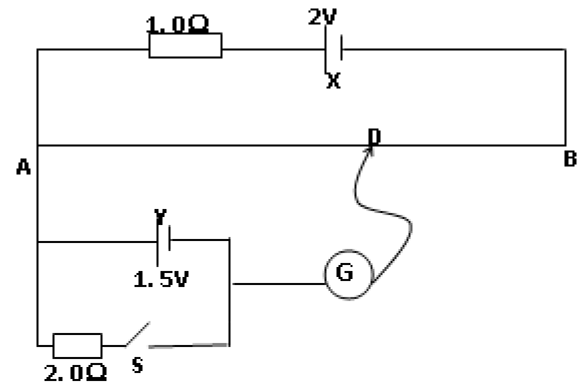
2. A dry cell gives a balance length of 84.8cm on a potentiometer wire. When a resistor of resistance 15Ω is connected across the terminals of the cell, a balance length of 75cm is obtained. Find the internal resistance of the cell.
$\frac{E}{V} = \frac{l}{l_1}$

$\frac{E}{V} = \frac{84.8}{75}, Where V = IR = \frac{ER}{R + r} = \frac{15E}{15 + r}$

$\frac{E}{\left(\frac{15E}{15+r}\right)} = \frac{84.8}{75}$

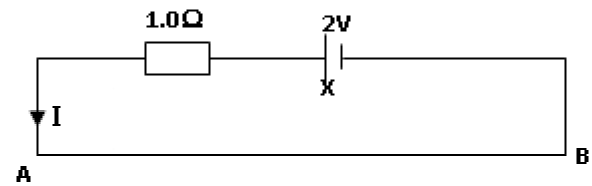
$\frac{15 + r}{15} = \frac{84.8}{75}$
$\boldsymbol{r = 1.96}$Ù

3. AB is a uniform wire of length 1m and resistance 4Ω. X is battery of emf 2V and negligible internal resistance. Y has emf 1. 5V.




(i) Find the balance length of AD when the switch S is open.
(ii) If the balance length is 75cm, when the switch S is closed, find the internal resistance of Y.

**Solution:**




(i)
Resistance of AB = 4Ω
Total resistance, =(4+1)= 5Ω

From the circuit formula;
$\mathrm{I} = \frac{\mathrm{E.m.f}}{\mathrm{Total\ resistance}}$

$\mathrm{I} = \frac{2}{5} = 0.4\ A$
Resistance per centimetre, $(R/cm)$ of AB is;

$(R_{AB}/cm) = \frac{R_{AB}}{l_{AB}} = \frac{4}{100} = 0.04\Omega m^{-1}$

Potential difference per centimetre, $(V/cm) = K$ of AB is;

$K = I(R_{AB}/cm)$
$K = 0.4(0.04)$
$K = 0.016Vcm^{-1}$

When S is open, then at balance;
$E_Y = Kl_{AD} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(i)$
$1.5 = 0.016l_{AD}$
$\boldsymbol{l_{AD} = 93.75cm}$

(ii) When S is closed, then at balance;
$V_R = Kl_{AD} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$

$$\frac{E_Y}{V_R} = \frac{l}{l_1}$$

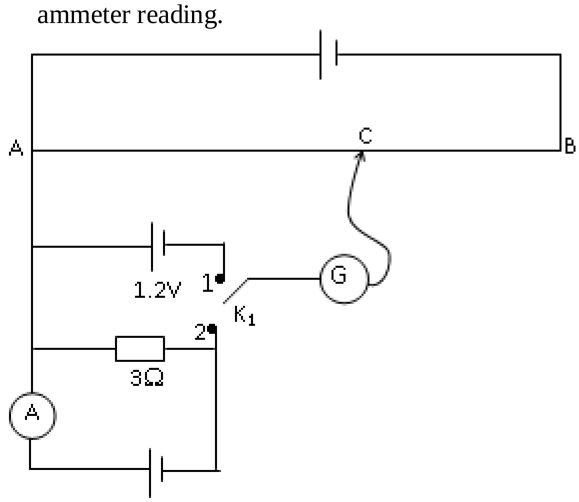
$$\frac{E_Y}{V_R} = \frac{93.75}{75}; Where V_R = IR = \frac{E_Y R}{R+r} = \frac{2E_Y}{2+r}$$

$$\frac{E_Y}{\left(\frac{2E_Y}{2+r}\right)} = \frac{93.75}{75}$$

$$\frac{2+r}{2} = \frac{93.75}{75}$$

$\boldsymbol{r = 0.5}$**Ù**

4. In the circuit below, when $K_1$ is connected to position 1, the balance length AC =30.2cm. When $K_1$ is connected to position 2, the balance length AC = 26.8cm and the ammeter reading is 0.4A. Find the percentage error in the ammeter reading.




**Solution**
When $K_1$ is in position 1,
$E_Y = Kl_{AC} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(i)$
$1.2 = 30.2K$
$K = \frac{12}{302} = \frac{6}{151} Vcm^{-1}$

When K1 is in position 2, the cell and 3Ω are in parallel.
$V = Kl_{AC} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$
$V = 26.8K$
$V = 26.8 \times \frac{6}{151} = 1.065V$

Current through the ammeter, I
$But\,; E = V_R = 1.065V, Since, E\ and\ 3\Omega\ are\ in\ parallel.$
$\Leftrightarrow\quad E = I(R)$
$1.065 = I(3)$
$1.065 = 3I \Leftrightarrow I = \frac{1.065}{3} = 0.355A$

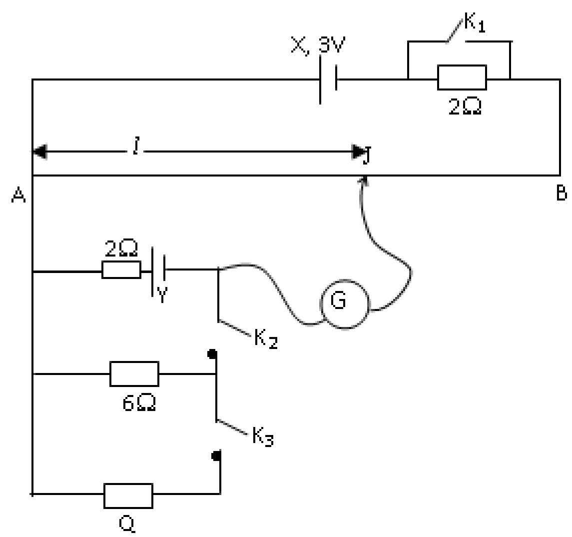
$Error = (0.4 - 0.355) = 0.045\ A$

$$Percentage\ error = \frac{Error}{Actual\ ammeter\ reading} \times 100\%$$

$$Percentage\ error = \frac{0.045}{0.355} \times 100\%$$

$Percentage\ error = 12.7\%$

5. In the figure below, X is an accumulator of e.m.f 3V and having negligible internal resistance. AB is a uniform slide wire 1.2m long and having a resistance of 4Ω. Y is a cell of un known e.m.f and internal resistance.

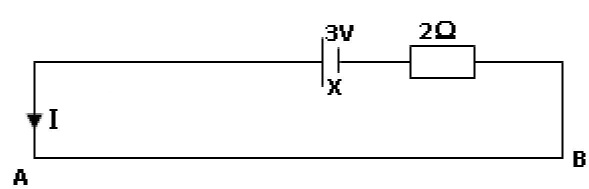



When $K_1$, $K_2$ and $K_3$ are open, G shows no deflection when AJ = 90cm. When $K_1$ and $K_2$ are closed with $K_3$ open, G shows no deflection when AJ = 43.9cm. When$K_2$ and $K_3$ are closed and $K_1$ left open, G shows no deflection when AJ = 58.1cm.

Find the;
(i). E.m.f of the cell Y
(ii). Internal resistance of cell Y.
(iii). Value of Q.
(iv). Balance length when $K_1$, $K_2$, and $K_3$ are all closed.

**Solution**

When $K_1$ is open, current passes through the 2Ω resistor.




(i)
Resistance of AB = 4Ω
Total resistance, =(4+2)= 6Ω

From the circuit formula;

$$I = \frac{\text{E.m.f}}{\text{Total resistance}}$$

$$I = \frac{3}{6} = 0.5\ A$$

Resistance per centimetre, $(R/cm)$ of AB is;

$$(R_{AB}/cm) = \frac{R_{AB}}{l_{AB}} = \frac{4}{120} = \frac{1}{30}\Omega m^{-1}$$

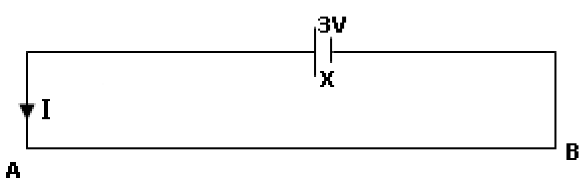
Potential difference per centimetre, $(V/cm) = K$ of AB is;

$K = I(R_{AB}/cm)$

$K = 0.5\left(\frac{1}{30}\right)$

$K = \frac{1}{60}Vcm^{-1}$

When $K_1$ is closed, current dodges the 2Ω resistor.




(i)

Resistance of AB = 4Ω

Total resistance, = 4Ω

From the circuit formula;

$$I = \frac{\text{E.m.f}}{\text{Total resistance}}$$

$$I = \frac{3}{4} = 0.75\ A$$

Resistance per centimetre, $(R/cm)$ of AB is;

$$(R_{AB}/cm) = \frac{R_{AB}}{l_{AB}} = \frac{4}{120} = \frac{1}{30}\Omega m^{-1}$$

Potential difference per centimetre, $(V/cm) = K$ of AB is;

$K = I(R_{AB}/cm)$

$K = 0.75\left(\frac{1}{30}\right)$

$K = \frac{1}{40} = 0.025Vcm^{-1}$

(i) At balance;

When $k_1$, $k_2$ and $k_3$ are open, then at balance;

$l_{AJ} = 90cm,\ K = \frac{1}{60}Vcm^{-1}$

$E_Y = Kl_{AJ}$ ………………………………………. $(i)$

$E_Y = \frac{1}{60} \times 90 = 1.5V$

$\boldsymbol{E_Y = 1.5V}$

Alternatively;

$E_Y = V_{AJ} = IR_{AJ} = I\left[(R/cm) \times l_{AJ}\right]$

$E_Y = I\left[(R/cm) \times l_{AJ}\right]$

$$E_Y = \left(\frac{E.m.f}{Totalresistance}\right)\left[\left(\frac{R_{AB}}{l_{AB}}\right) \times l_{AJ}\right]$$

$$E_Y = \left(\frac{3}{6}\right)\left[\left(\frac{4}{120}\right) \times 90\right]$$

$\boldsymbol{E_Y = 1.5V}$

(ii) When $K_1$ and $K_2$ are closed and $K_3$ open, then at balance;

$l_{AJ} = 43.9cm,\ K = \frac{1}{40}Vcm^{-1}$

$V_{AJ} = IR = Kl_{AJ}$ ……………………………………… $(ii)$

From the circuit formula;

$$I = \frac{E.m.f}{Total\ resistance} = \frac{E_Y}{6+2+r} = \frac{E_Y}{8+r}$$

$$V_{AJ} = IR = \frac{E_YR}{R+r} = \frac{6E_Y}{6+2+r} = \frac{6E_Y}{8+r}$$

$$\frac{6E_Y}{8+r} = Kl_{AJ}$$

$$\frac{6 \times 1.5}{8+r} = \frac{1}{40} \times 43.9$$

$$\frac{9}{8+r} = 1.0975$$

$8.78 + 1.0975r = 9$

$\boldsymbol{r = 0.2}$Ù

(iii) When $K_1$ is open, $K_2$ and $K_3$ are closed, then at balance;

$l_{AJ} = 58.1cm,\ K = \frac{1}{60}Vcm^{-1}$

$V_{AJ} = IR = Kl_{AJ}$ …………………………………… $(iii)$

$Lettotalresistance = R_p$

$\frac{1}{R_p} = \frac{1}{Q} + \frac{1}{6}$ ……………………………………… $(iv)$

From the circuit formula;

$$I = \frac{E.m.f}{Total\ resistance} = \frac{E_Y}{R_p+2+r}$$

$$V_{AJ} = IR_p = \frac{E_YR_p}{R_p+2+r} = \frac{1.5R_p}{R_p+2+0.2} = \frac{1.5R_p}{R_p+2.2}$$

$$\frac{1.5R_p}{R_p+2.2} = Kl_{AJ}$$

$$\frac{1.5R_p}{R_p + 2.2} = \frac{1}{60} \times 58.1$$

$$\frac{1.5R_p}{R_p + 2.2} = \frac{581}{600}$$

$900R_p = 581R_p + 1278.2$
$R_p = 4.0\Omega$
$\boldsymbol{R_p = 4.0}$Ù

Substitute for $R_p$ in equation (iv)
$$\frac{1}{R_p} = \frac{1}{Q} + \frac{1}{6}$$
$$\frac{1}{4} = \frac{1}{Q} + \frac{1}{6} \Leftrightarrow \frac{1}{Q} = \frac{1}{4} - \frac{1}{6} = \frac{1}{12} \Leftrightarrow Q = 12\Omega$$
$\boldsymbol{Q = 12}$Ù
(iv) When K$_1$, K$_2$ and K$_3$ are closed, then at balance;
$l_{AJ} = ?,\ K = \frac{1}{40} Vcm^{-1}$

$V_{AJ} = IR = Kl_{AJ} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (iii)$

From the circuit formula;
$$I = \frac{E.m.f}{Total\ resistance} = \frac{E_Y}{R + 2 + r}$$

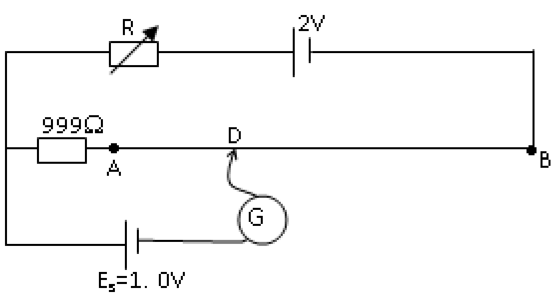
$$V_{AJ} = IR_p = \frac{E_Y R}{R + 2 + r} = \frac{1.5 \times 4}{4 + 2 + 0.2} = \frac{6}{6.2}$$
$$\frac{6}{6.2} = Kl_{AJ}$$

$$\frac{6}{6.2} = \frac{1}{40} \times\ l_{AJ}$$

$$\frac{6}{6.2} = \frac{l_{AJ}}{40}$$

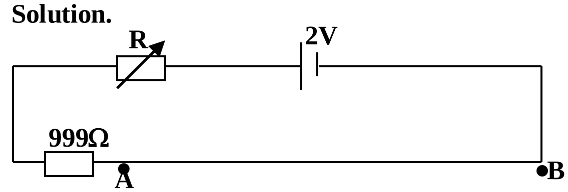
$\boldsymbol{l_{AJ} = 38.71cm}$

6.




The slide wire has length 100cm and resistance 10Ω. The galvanometer shows no deflection when AD = 10cm. Find
(i) the current flowing in the driver cell
(ii) the value of R.
(iii) the emf of thermocouple which balanced by a length of 60cm of the slide AB.

**Solution.**




(i)
Resistance of AB = 10Ω
Total resistance, = (10 + 999 + R)Ω = (1009+R)Ω

$E_s = IR_{999} + IR_{AD} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (i)$
$E_s = 999I + I[(R/cm) \times l_{AD}]$
$$1 = 999I + I\left[\left(\frac{10}{100}\right) \times 10\right]$$
$1 = 1000I$
$$\boldsymbol{I = \frac{1}{1000}\ A = 0.001A}$$
From the circuit formula;
$$I = \frac{E.m.f}{Total\ resistance}$$

$$0.001 = \frac{2}{1009 + R}$$
$0.001(1009 + R) = 2$
$.\boldsymbol{R = 991}$Ù

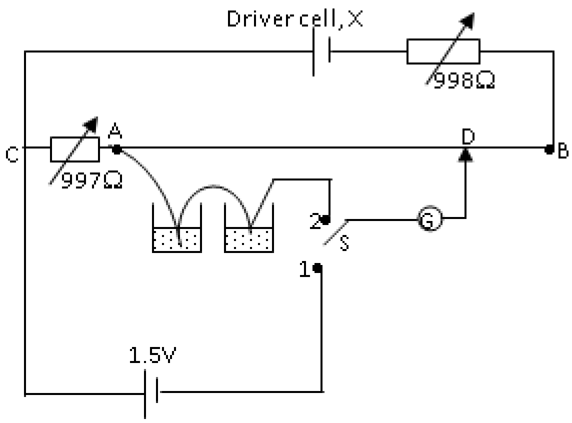
Resistance per centimetre, $(R/cm)$ of AB is;
$$(R_{AB}/cm) = \frac{R_{AB}}{l_{AB}} = \frac{10}{100} = \frac{1}{10} = 0.1\Omega m^{-1}$$

Potential difference per centimetre, $(V/cm) = K$ of AB is;
$K = I(R_{AB}/cm)$
$$K = 0.001\left(\frac{1}{10}\right)$$
$$K = \frac{1}{10000} = 0.00001Vcm^{-1}$$

(ii) At balance; The e.m.f of the thermocouple, when the cell E$_s$ is replaced by a thermocouple of e.m.f E$_T$.
$l_{AB} = 60cm,\ K = 0.0001Vcm^{-1}$

$E_T = Kl_{AD} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (ii)$
$E_T = 0.0001 \times 60$
$\boldsymbol{E_T = 0.006V = 6mV}$

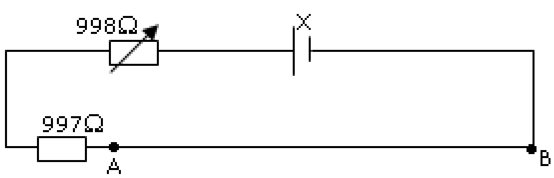
7. In the figure, below, AB is a uniform slide wire of resistance 5Ω. X is a cell of unknown e.m.f and having negligible internal resistance.




When S is put in position 1, G shows no deflection when AD =60cm. When S is placed in position 2, a balance length of 40.0cm is obtained. Find the;

(i). Current flowing in the slide wire.
(ii). E.m.f of cell X.
(iii). E.m.f of the thermocouple.

**Solution.**




(i)
Resistance of AB = 5Ω
Total resistance, = (5 + 997 + 998)Ω = 2000Ω
From the circuit formula;

$$I = \frac{E.m.f}{Total\ resistance}$$

$$I = \frac{E_X}{2000} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(i)$$

Resistance per centimetre, $(R/cm)$ of AB is;

$$(R_{AB}/cm) = \frac{R_{AB}}{l_{AB}} = \frac{5}{100} = \frac{1}{20} = 0.05\Omega m^{-1}$$

Potential difference per centimetre, $(V/cm) = K$ of AB is;
$K = I(R_{AB}/cm)$

$$K = \left(\frac{E_X}{2000}\right)\left(\frac{1}{20}\right) = \frac{E_X}{40000} Vcm^{-1}$$

At balance; when S is in position 1, the cell $E_s$ of 1.5V is balanced by lengths CA and AB.

$l_{AD} = 60cm,\ K = \frac{E_X}{40000}\ Vcm^{-1}$

$$E_s = IR_{CA} + Kl_{AD} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(ii)$$
$$1.5 = 997I + \left(\frac{E_X}{40000}\right) \times 60$$
$$1.5 = 997I + \left(\frac{2000I}{40000}\right) \times 60$$
$$1.5 = 997I + \left(\frac{I}{20}\right) \times 60$$
$$1.5 = 997I + 3I$$
$$1.5 = 1000I$$
$$\boldsymbol{I} = \frac{1.5}{1000} = \mathbf{1.5 \times 10^{-3}}\ \boldsymbol{A}.$$

(ii) E.m.f of cell X
Substituting for I in equation (i)
$E_X = 2000I$
$E_X = 2000 \times (1.5 \times 10^{-3})$
$\boldsymbol{E_X = 3V}$

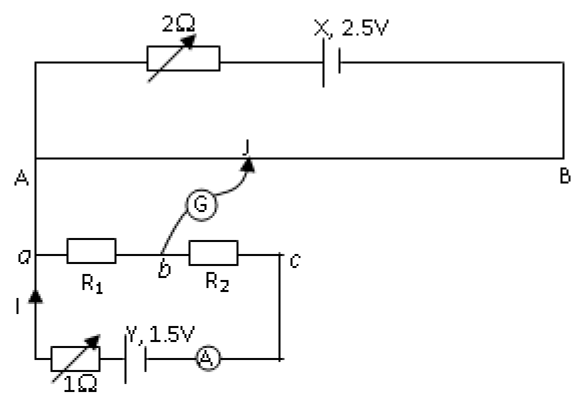
(iii) At balance when S is in position 1, the cell $E_s$ of 1.5V is balanced by lengths CA and AB.
$l_{AD} = 40cm,\ K = \frac{E_X}{40000} = \frac{3}{40000}\ Vcm^{-1}$

$$E_T = Kl_{AD} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(ii)$$

$$E_T = \left(\frac{3}{40000}\right) \times 40$$
$$\boldsymbol{E_T = 0.003V or 3mV}$$

Alternatively;
$E_T = V_{AD} = IR_{AD}$
$E_T = V_{AD} = 1.5 \times 10^{-3}\big((R_{AB}/cm) \times l_{AD}\big)$
$E_T = V_{AD} = 1.5 \times 10^{-3}(0.05 \times 40)$
$\boldsymbol{E_T = 0.003V or 3mV}$

8. In the figure below, X is an accumulator of e.m.f 2.5V and having negligible internal resistance. Y is a cell of e.m.f 1.5V and having internal resistance of 1Ω.

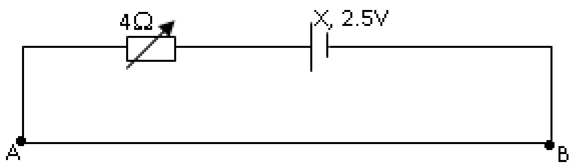



AB is uniform slide wire 1m long and having 40 Ohms. With connections at **a** and **b**, a balance length of 39.8cm was obtained. When the connections at a and b are replaced by those at **a** and **c**, a balance length of 82.4cm was obtained. Find the;

(i) Current flowing through the slide wire

(ii) Resistances $R_1$ and $R_2$

**Solution.**




(i)

Resistance of AB = 2Ω

Total resistance, = (4+2)Ω = 6Ω

From the circuit formula;

$$I = \frac{E.m.f}{Total\ resistance}$$

$$I = \frac{2.5}{6} = \frac{5}{12}\ A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$$

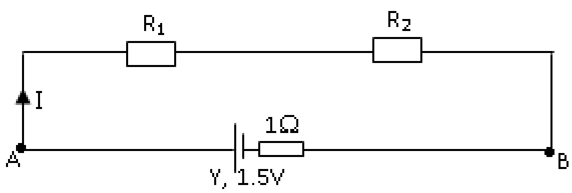
Resistance per centimetre, $(R/cm)$ of AB is;

$$(R_{AB}/cm) = \frac{R_{AB}}{l_{AB}} = \frac{4}{100} = \frac{1}{25} = 0.04\Omega m^{-1}$$

Potential difference per centimetre, $(V/cm) = K$ of AB is;

$K = I(R_{AB}/cm)$

$$K = \left(\frac{5}{12}\right)\left(\frac{1}{25}\right) = \frac{1}{60}Vcm^{-1}$$




(i)

Total resistance, = $(1+R_1+R_2)\Omega$

From the circuit formula;

$$I = \frac{E.m.f}{Total\ resistance}$$

$$I = \frac{1.5}{1 + R_1 + R_2}$$

$I(1 + R_1 + R_2) = 1.5$

$I + I(R_1 + R_2) = 1.5 \ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$

With connections at **a** and **b**, At balance;

$l_{AJ} = 39.8cm,\ K = \frac{1}{60}Vcm^{-1}$

$IR_1 = Kl_{AJ}$

$IR_1 = \frac{1}{60} \times 39.8$

$IR_1 = 0.6633 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(ii)$

With connections at **a** and **c**, At balance;

$l_{AJ} = 82.4cm,\ K = \frac{1}{60}Vcm^{-1}$

$I(R_1 + R_2) = Kl_{AJ}$

$I(R_1 + R_2) = \frac{1}{60} \times 82.4$

$I(R_1 + R_2) = 1.3733 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots\ldots.(iii)$

From equation (i) and (iii)

$I + I(R_1 + R_2) = 1.5$

$I + 1.3733 = 1.5$

$$\boldsymbol{I = 0.1267A}$$

From equation (ii)

$IR_1 = 0.6633$

$0.1267R_1 = 0.6633$

$\boldsymbol{R_1 = 5.24}$Ù

From equation (iii)

$I(R_1 + R_2) = 1.3733$

$0.1267(5.24 + R_2) = 1.3733$

$\boldsymbol{R_2 = 5.6}$Ù

**Exercise**

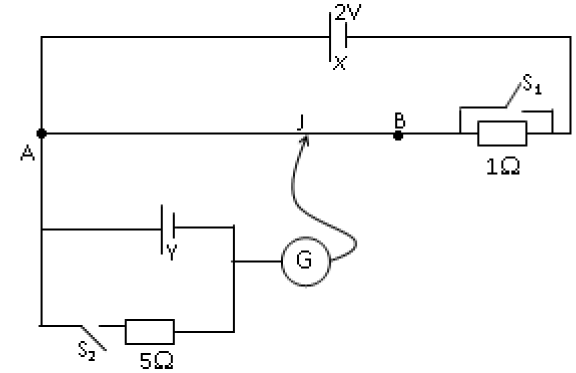
1. A 1Ω resistor is in series with an ammeter m in a circuit. The p.d across the resistor is balanced by a length of 60cm on a potentiometer wire. A standard cell of emf 1.02V is balanced by a length of 50cm. If m reads 1.1A, what is the error in the reading? (0.124A)

2. A potentiometer wire of length 1m and resistance 1Ω is used to measure an emf of the order mV. A battery of emf 2V and negligible internal resistance is used as a driver cell. Calculate the resistance to be in series with potentiometer so as to obtain a potential drop of 5mV across the wire. (399Ω)

3. In a potentiometer, a cell of emf **x** gave a balance length of **a** cm and another cell of emf **y** gave a balance length of **b** cm. When the cells are connected in series, a balance length of **c** cm was obtained. It was also discovered that **a + b ≠ c**.Show that the true ratio: $\frac{x}{y} = \frac{c-b}{c-a}$.

4. In the circuit below, ***x*** has negligible internal resistance and length AB is 100cm and resistance of AB is 5Ω.when

$S_1$ and $S_2$ are open, the balance length AJ = 90cm. When $S_2$ is closed and $S_1$ open, the balance length AJ = 75cm..



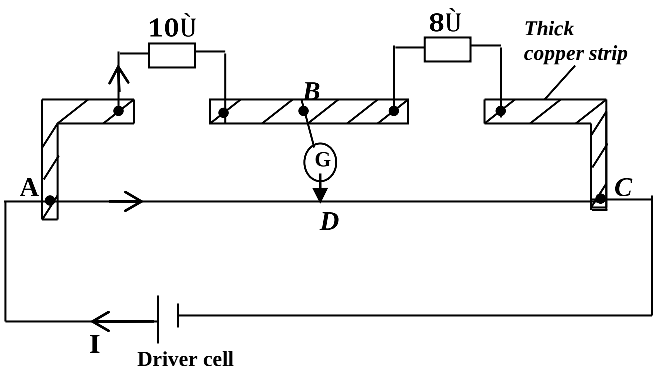

Find the;

(i) emf of cell y (1.5V)

(ii) Internal resistance of cell y. (1Ω)

(iii) Balance length when $S_1$ and $S_2$ are closed. (62.5cm)

5.




In the figure above, the 10Ω resistor is shunted by a short length of a wire. The balance length is found to be 0.32m from the left hand end of the metre-bridge.

(i). Calculate the resistance of the shunt. [$R_s = 6\Omega$]

(ii). If the shunt has a radius of $9.01 \times 10^{-3} m$ and is160m long, calculate its resistivity.

[$ñ = 9.56 \times 10^{-4} \Omega m$]

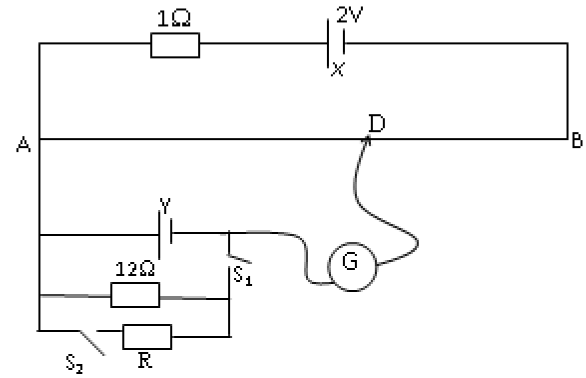
6. The resistivity of mild steel is $1.5 \times 10^{-7} \Omega m$ at 20$^0$C and its temperature coefficient of resistance is $5.0 \times 10^{-4} K^{-1}$. Calculate its resistivity at 60$^0$C. [$ñ_{60} = 1.733 \times 10^{-7} \Omega m$].

7. In a potentiometer experiment, the balance length of a wire is 53.4cm when a standard cell is of 1.08V is used. The balance length with cell X is 72.6cm.

(i). Draw a complete sketch of the circuit arrangement.

(ii). Calculate the e.m. f of the cell X. [$E_X = 1.49V$ ] .

8.



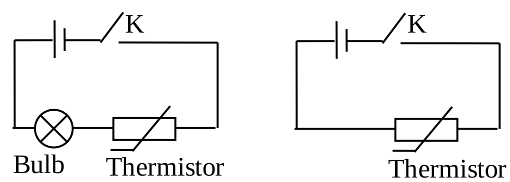

Length AB is 1 metre and has a resistance of 4Ω. Y is a cell of un known e.m.f and internal resistance, when $s_1$ and $s_2$ are open, the balance length is 93.8 cm. When $s_2$ only is open, the balance length is 86.5cm. When both $s_1$ and $s_2$ are closed, the balance length is 75cm. Calculate the;

(i). E.m.f of cell Y.[$E = 1.5V$ ]

(ii). Internal resistance, r of cell Y. [$r = 1.01\Omega$ ]

(iii). Value of R. [$R = 6\Omega$ ]

9. A cell, a bulb, a switch, k and a thermistor with negative temperature coefficient of resistance are connected as shown below.



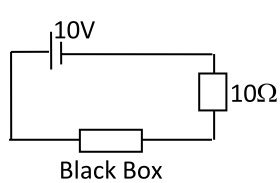

(a) (b)

(i) Explain what would happen when switch K, in figure (a) above is closed.

(ii) If the bulb in (a) is removed and the circuit connected as in figure (b), explain what would happen when switch K is closed.

10. (a) A steady current of I amperes is maintained in a resistor of resistance R ohms for t seconds. Derive an expression for the thermal energy dissipated in R in terms of I, R, and t.

(b) In the figure, a 10V d.c supply is connected in series with a resistor of 10Ω and a black box containing a source of e.m.f and another electrical component.




If the p.d across the 10Ω is 5V, and that energy is dissipated in the black box as thermal energy at a rate of 1.5W, calculate the;

(i) Other electrical component in the black box.

(ii) E.m.f and polarity of the source in the black box.
(Ans: E.m.f =2V)

11. An 80.0cm copper wire of diameter 1.0mm is joined end to end with a 50.0cm iron wire of the same diameter to form a loop. A current of 4A is fed into the loop. Find the;
    (i) Effective resistance of the loop.
    (ii) Electric field intensity in the copper wire.
    (iii) Current through the copper wire

[Resistivities of copper and Iron respectively are;
$1.7 \times 10^{-8} \Omega m and 1.0 \times 10^{-7} \Omega m$]

12. See UNEB: 1998/2 No. 8 (b)
    1999/2 No. 8 (d)
    2001/2 No. 10(d)
    2004/2 No. 10
    2008/2 No. 8